Discrete Applied Mathematics 201 (2016) 201–212

Contents lists available at ScienceDirect

Discrete Applied Mathematics

journal homepage: www.elsevier.com/locate/dam

On orthogonal ray trees

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ARTICLE INFO

Article history:
Received 27 February 2015
Received in revised form 25 July 2015
Accepted 31 July 2015
Available online 24 August 2015

Keywords:
Edge-asteroids
Forbidden minors
Intersection graphs
Orthogonal ray graphs
Orthogonal ray trees

ABSTRACT

An orthogonal ray graph is an intersection graph of horizontal rays (closed half-lines) and vertical rays in the plane, which is introduced in connection with the defect-tolerant design of nano-circuits. An orthogonal ray graph is a 3-directional orthogonal ray graph if every vertical ray has the same direction. A 3-directional orthogonal ray graph is a 2-directional orthogonal ray graph if every horizontal ray has the same direction. The characterizations and the complexity of the recognition problem have been open for orthogonal ray graphs and 3-directional orthogonal ray graphs, while various characterizations with a quadratic-time recognition algorithm have been known for 2-directional orthogonal ray graphs. In this paper, we show several characterizations with a linear-time recognition algorithm for orthogonal ray trees by using the 2-directional orthogonal ray trees. We also show that a tree is a 3-directional orthogonal ray graph if and only if it is a 2-directional orthogonal ray graph. Moreover, we show some necessary conditions for orthogonal ray graphs and 3-directional orthogonal ray graphs.

1. Introduction

A graph G is called an *intersection graph* if there exists a set of objects such that each vertex corresponds to an object and two vertices are adjacent if and only if the corresponding objects intersect. Such a set of objects is called a *representation* of G. Intersection graphs of geometric objects have been extensively investigated, since the representations allow us to design efficient algorithms. The intersection graphs of geometric objects have many applications in various areas including integrated circuits, scheduling, and bioinformatics. See [3,11,18,28] for survey.

Segment graphs are the intersection graphs of straight-line segments in the plane, and one of the most natural and well-studied classes of the intersection graphs [5,16]. The recognition problem for segment graphs is known to be NP-hard [17]. A segment graph is called a *grid intersection graph* [1,12] if the lines are restricted to being parallel to the x- and y-axes (horizontal and vertical) such that no two parallel segments intersect. The recognition problem for grid intersection graphs is also known to be NP-complete [15]. A grid intersection graph is called a *unit grid intersection graph* [21] if every line segments have the same (unit) length. Recently, it has been shown in [20] that the recognition problem for unit grid intersection graphs is NP-complete.

Besides the segment graphs, the intersection graphs of rays (closed half-lines) in the plane have been considered [4,14,25]. We focus on the case where every rays are parallel to the x- and y-axes. Such intersection graphs are called orthogonal ray

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http://dx.doi.org/10.1016/j.dam.2015.07.034

graphs [25]. Formally, a bipartite graph G with bipartition (U, V) is called an *orthogonal ray graph* (*ORG* for short) if there exist a set of disjoint horizontal rays R_u, $u \in U$, in the xy-plane, and a set of disjoint vertical rays R_v, $v \in V$, such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. A set $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$ is called an *orthogonal ray representation* of G. The ORGs are introduced in connection with the defect-tolerant design of nano-circuits [24]. An ORG G with bipartition (U, V) is called a *3-directional orthogonal ray graph* (*3-DORG* for short) if G has an orthogonal ray representation $\mathcal{R}(G)$ such that every vertical ray $R_v \in \mathcal{R}(G)$, $v \in V$, has the same direction. An ORG G with bipartition (U, V) is called a *2-directional orthogonal ray graph* (*2-DORG* for short) if G has an orthogonal ray representation $\mathcal{R}(G)$ such that every horizontal ray $R_u \in \mathcal{R}(G)$, $u \in U$, has the same direction and every vertical ray $R_v \in \mathcal{R}(G)$, $v \in V$, has the same direction.

Among the graph classes above, the following relationship has been known [25]: {2-Directional Orthogonal Ray Graphs} $\subset$ {Orthogonal Ray Graphs} $\subset$ {Unit Grid Intersection Graphs} $\subset$ {Grid Intersection Graphs} $\subset$ {Bipartite Graphs}, where $X \subset Y$ indicates a set X is a proper subset of Y.

The 2-DORGs have been well investigated [8,22–27,30,31], and various characterizations have been known [25,30]. One of the characterizations is that 2-DORGs are the complements of circular-arc graphs with clique cover number 2, which is a well-studied class of graphs [9,13,29,32]. Based on the characterization, 2-DORGs can be recognized in $O(n^2)$ time, where n is the number of vertices in a graph. The 2-DORGs also has a forbidden graph characterization such that a bipartite graph is a 2-DORG if and only if it contains no induced cycle of length at least 6 or edge-asteroids [9,25].

On the other hand, the characterizations and the complexity of the recognition problem have been open for ORGs and 3-DORGs. As the first step to understand ORGs and 3-DORGs, it is natural to study the case of trees. A tree is called an *orthogonal ray tree* (*ORT* for short) if it is an orthogonal ray graph. An ORT is called a *3-directional orthogonal ray tree* (*3-DORT* for short) if it is a 3-DORG, and called a *2-directional orthogonal ray tree* (*2-DORT* for short) if it is a 2-DORG. The 2-DORTs have been investigated, and several characterizations with a linear-time recognition algorithm are known [24,25]. We have also known that any tree is a unit grid intersection graph [21].

The purpose of the paper is to show several characterizations with a linear-time recognition algorithm for ORTs and 3-DORTs by using the characterizations of 2-DORTs. We also show some necessary conditions for ORGs and 3-DORGs.

We show in Section 2 some characterizations for 2-DORGs and 2-DORTs used in this paper. In Section 3, we introduce a new forbidden structure, an asteroidal quintuple of edges (A5E for short), and show that any ORG contains no A5Es, which is also a sufficient condition for ORTs as shown in Section 4. We also show in Section 4 that any ORT is a graph obtained from two 2-DORTs by identifying a vertex in one 2-DORT with a vertex in the other. Moreover, we show a forbidden minor characterization with a linear-time recognition algorithm for ORTs. In Section 5, we show that any 3-DORG contains no edge-asteroids, and hence, a tree is a 3-DORT if and only if it is a 2-DORT.

The characterizations and the complexity of the recognition problem for ORGs and 3-DORGs still remain interesting open questions.

2. Two-directional orthogonal ray graphs

We show in this section some preliminaries and several characterizations for 2-DORGs and 2-DORTs used in this paper. See [24,25] for more information.

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. The *open neighborhood* of a vertex v of G is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E(G)\}$, and the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. For an edge $e = (u, v)$ of G, we use $N_G[e]$ to denote the set of vertices adjacent to u or v, that is, $N_G[e] = N_G[u] \cup N_G[v]$. If no confusion arises, we will omit the index G.

A bipartite graph is called a *chordal bipartite graph* if it contains no induced cycles of length at least 6.

Let P be a path of length k with $V(P) = \{v_0, v_1, \ldots, v_k\}$ and $E(P) = \{e_1, e_2, \ldots, e_k\}$, where $e_i = (v_{i-1}, v_i)$, $1 \le i \le k$. We refer to P as a path from e_1 to e_k. A set of edges $\{e_0, e_1, \ldots, e_{2k}\} \subseteq E(G)$, $k \ge 1$, of a graph G is called an *edge-asteroid* of size $2k+1$ if for any i, $0 \le i \le 2k$, there exists a path from e_i to e_{i+1} that contains no vertices in $N[e_{i+k+1}]$ (subscripts are modulo $2k+1$). See Fig. 1 for examples of edge-asteroids. Edge-asteroids are introduced in [9], and 2-DORGs can be characterized as follows.

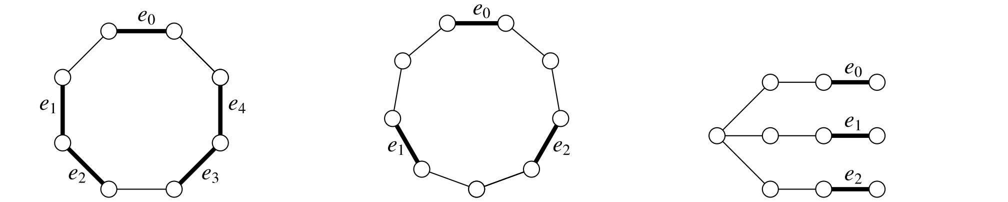
Theorem A ([9,25]). *A bipartite graph is a 2-DORG if and only if it is a chordal bipartite graph and contains no edge-asteroids.* □

The graph obtained from a complete bipartite graph $K_{1,3}$ (which is also known as a *claw*) by replacing each edge with a path of length 3 is called a *3-claw*. The 3-claw contains an edge-asteroid as shown in Fig. 1(c). A path P in a tree T is called a *spine* of T if every vertex of T is within distance 2 from a vertex on P. It has been known that 2-DORTs can be characterized as follows.

Theorem B ([24,25]). *The following statements are equivalent for a tree T:*

(i) *T is a 2-DORT;*
(ii) *T contains no 3-claw as a subtree;*
(iii) *T has a spine.* □

Theorem B implies a linear-time recognition algorithm for 2-DORTs [25], since it suffices to verify whether a longest path in a given tree is a spine, and a longest path in a tree can be obtained in linear time [7].



(a) An edge-asteroid of size 5 in C_8. (b) An edge-asteroid of size 3 in C_9. (c) An edge-asteroid of size 3 in a 3-claw.

Fig. 1. Examples of edge-asteroids, denoted by bold lines.

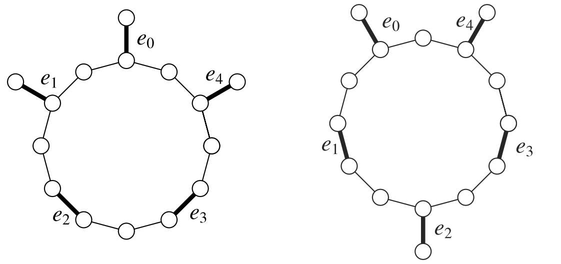


Fig. 2. Examples of A5Es, denoted by bold lines.

3. Orthogonal ray graphs

Although we have no characterizations for orthogonal ray graphs, two necessary conditions have been known. The first condition is included in [25].

Theorem C ([25]). *A cycle C_{2n} of length $2n$ is an ORG if and only if $2 \leq n \leq 6$.* □

The second condition is a forbidden structure similar to edge-asteroid [9], asteroidal triple of edges [19], and edge-asteroidal set [6]. The following arguments are appeared in [23]. A set of five edges $\{e_0, e_1, \ldots, e_4\} \subseteq E(G)$ of a graph G is called an *asteroidal quintuple of edges* (*A5E* for short) if for any i, $0 \leq i \leq 4$, there exists a path from e_i to e_{i+1} that contains no vertices in $N[e_{i-1}] \cup N[e_{i+2}]$ (subscripts are modulo 5). Examples of A5Es are shown in Figs. 2 and 5. The following is immediate from the definition of A5Es.

Lemma D ([23]). *If e_i and e_j are distinct edges in an A5E of a graph, then e_i and e_j share no common vertex and they are not joined by an edge.* □

The following theorem has also appeared in [23]. We include a proof here.

Theorem E ([23]). *Any ORG contains no A5Es.* □

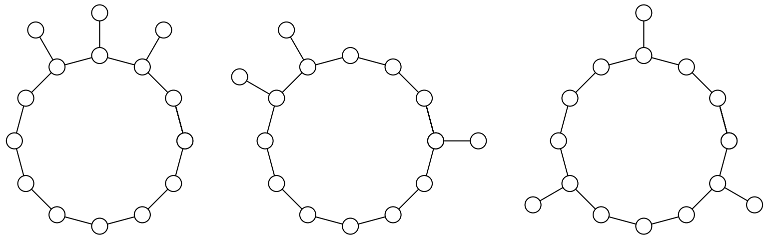
Proof. Let G be an ORG with bipartition (U, V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$. For the representation $\mathcal{R}(G)$, each edge of G can be classified into four types as up-right, down-right, up-left, or down-left, depending on the orientations of the horizontal ray (rightward or leftward), and the vertical ray (upward or downward) corresponding to the end-vertices of the edge.

We prove the theorem by contradiction. Suppose that G contains an A5E $E_5 = \{e_0, e_1, \ldots, e_4\}$. We have from Lemma D that the rays corresponding to the end-vertices of the edges of E_5 are distinct and disjoint, except for the intersection corresponding to the edges of E_5. Since $|E_5| = 5$, at least two edges in E_5 have the same type. We assume without loss of generality that edges e_i and e_j are both of type up-right. For an up-right edge $e = (u, v)$ with $u \in U$ and $v \in V$, two rays R_u and R_v divide the plane into two regions. We refer to the region above R_u and on the right side of R_v as the *inner* region of e, and the other as the *outer* region. We further assume that the rays corresponding to the end-vertices of e_i lie in the inner region of e_j, and if E_5 has other up-right edges then the rays of the up-right edges lie in the outer region of e_j. The rays of the edges in E_5 of the other type lie in the outer region of e_j, since they do not intersect with the rays corresponding to the end-vertices of e_j. Hence, any path from e_i to another edge in E_5 must have a vertex adjacent to at least one of the end-vertices of e_j.

We distinguish four cases: (i) If $j = i + 1$, let $k = i - 1$, (ii) if $j = i - 1$, let $k = i + 1$, (iii) if $j = i + 2$, let $k = i + 1$, and (iv) if $j = i - 2$, let $k = i - 1$ (additions and subtractions are modulo 5). In each case, any path from e_i to e_k must have a vertex adjacent to the end-vertices of e_j, contradicting to the definition of A5Es. Hence, G contains no A5Es. □

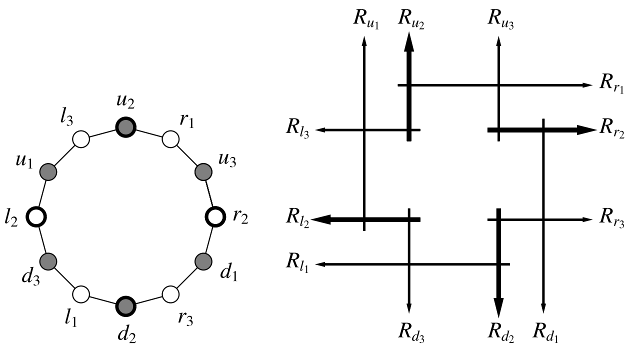
From the theorems above, we have the following.

Corollary F ([23]). *Any ORG contains no induced cycles of length at least 14 or A5Es.* □



(a) A graph C_{12_1}. (b) A graph C_{12_2}. (c) A graph C_{12_3}.

Fig. 3. Graphs with no induced cycle of length at least 14 or A5Es that is not ORG.



(a) A cycle C_{12}. (b) The representation of C_{12}.

Fig. 4. A cycle C_{12} and its orthogonal ray representation.

In the rest of this section, we show that the necessary condition in Corollary F is not sufficient. The characterization of ORGs remains an open question.

Theorem 1. *Graphs C_{12_1}, C_{12_2}, and C_{12_3} in Fig. 3 are not ORGs, while they contain no induced cycles of length at least* 14 *or A5Es.*

Proof. It is not difficult to see from Lemma D that the graphs contain no induced cycles of length at least 14 or A5Es. Now, we show that the graphs have no orthogonal ray representations. To prove this, we first consider the orthogonal ray representation $\mathcal{R}(C_{12})$ of C_{12}, the cycle of length 12. We use the following lemma.

Lemma G ([14]). *In an orthogonal ray representation of a cycle, at most three rays have the same direction.* □

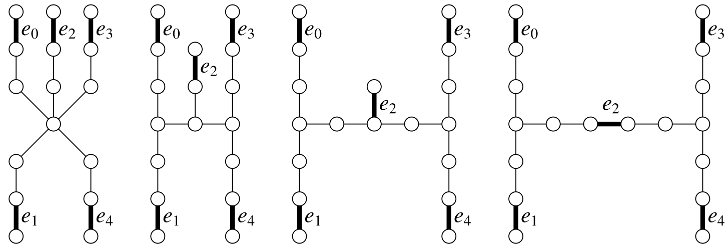
Hence, $\mathcal{R}(C_{12})$ has exactly three rays with the same direction. Let R_{u_1}, R_{u_2}, and R_{u_3} be the upward rays in $\mathcal{R}(C_{12})$ numbered from left to right (see Fig. 4(b) for example). Similarly, let R_{r_1}, R_{r_2}, and R_{r_3} be the rightward rays in $\mathcal{R}(C_{12})$ numbered from top to bottom, let R_{d_1}, R_{d_2}, and R_{d_3} be the downward rays in $\mathcal{R}(C_{12})$ numbered from right to left, and let R_{l_1}, R_{l_2}, and R_{l_3} be the leftward rays in $\mathcal{R}(C_{12})$ numbered from bottom to top. We number the corresponding vertices of C_{12} in the same way.

We can easily see that both R_{u_1} and R_{u_2} intersect R_{l_3} and both R_{u_2} and R_{u_3} intersect R_{r_1}. Similarly, both R_{r_1} and R_{r_2} intersect R_{u_3}, both R_{r_2} and R_{r_3} intersect R_{d_1}, both R_{d_1} and R_{d_2} intersect R_{r_3}, both R_{d_2} and R_{d_3} intersect R_{l_1}, both R_{l_1} and R_{l_2} intersect R_{d_3}, both R_{l_2} and R_{l_3} intersect R_{u_1}. Thus, the vertices in C_{12} corresponds to the rays as shown in Fig. 4(a) up to rotations and reflections.

Since R_{u_2} is in between $R_{u_1} \cup R_{d_3}$ and $R_{u_3} \cup R_{d_1}$, any ray intersecting R_{u_2} must intersect at least one of them. Thus, the graph obtained from C_{12} by joining a degree-1 vertex to u_2 is not an ORG. Similarly, the graph obtained from C_{12} by joining a degree-1 vertex to either r_2, d_2, or l_2 is not an ORG. Since the graph C_{12_1} is obtained from C_{12} by joining three degree-1 vertices to the consecutive three vertices, one degree-1 vertex must be joined to the vertex v_2, $v \in \{u, r, d, l\}$. Hence, C_{12_1} is not an ORG. Similarly, since one degree-1 vertex in C_{12_2} and C_{12_3} must be joined to v_2, $v \in \{u, r, d, l\}$, C_{12_2} and C_{12_3} are not ORGs. Then, we have the theorem. □

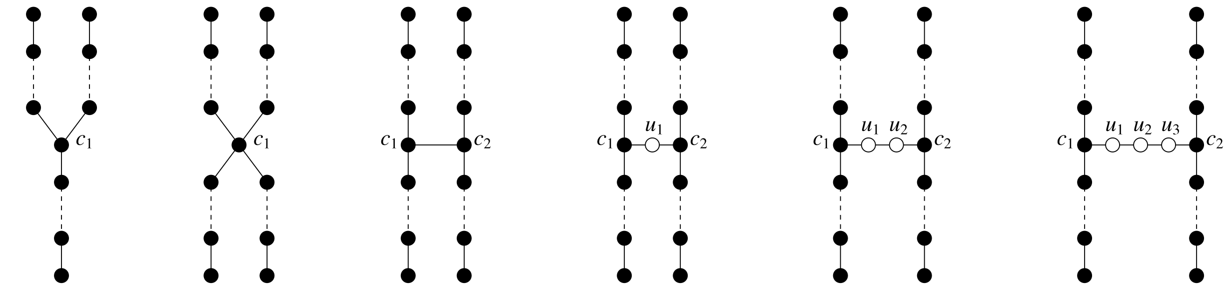
4. Orthogonal ray trees

In the previous section, we show that any ORG contains no A5Es, but this necessary condition is not sufficient to characterize ORGs. In this section, we show that the condition is sufficient for ORTs. We also show other characterizations with a linear-time recognition algorithm for ORTs.



(a) Tree F_0. (b) Tree F_1. (c) Tree F_2. (d) Tree F_3.

Fig. 5. The minimal list $\mathcal{F}$ of forbidden minors for ORTs. Bold edges denote A5Es.



(a) N_Y-type. (b) N_0-type. (c) N_1-type. (d) N_2-type. (e) N_3-type. (f) N_4-type.

Fig. 6. N_i-type trees, $i \in \{Y, 0, 1, \ldots, 4\}$.

4.1. Characterizations

The *contraction* of an edge $e = (u, v)$ of a graph G is the replacement of u and v with a new vertex w such that w is adjacent to the vertices in $(N_G(u) \cup N_G(v)) \setminus \{u, v\}$. A graph H is called a *minor* of a graph G if H is obtained from G by vertex deletions, edge deletions, and edge contractions.

The *identification* of a vertex u of a tree T_1 with a vertex v of a tree T_2 denotes the construction of a tree T from T_1 and T_2 by replacing u and v with a new vertex w such that w is adjacent to all the vertices in $N_{T_1}(u)$ and $N_{T_2}(v)$. The *splitting* a tree T at a vertex w of T is the reverse operation of the identification of two vertices, that is, it produces a pair of trees T_1 and T_2 by replacing w with a pair of new vertices $u \in V(T_1)$ and $v \in V(T_2)$ such that each vertex in $N_T(w)$ is adjacent to either u or v.

Let $\mathcal{F}$ be the set of trees shown in Fig. 5. We can see from the figures that every tree in $\mathcal{F}$ contains an A5E. It turns out in Theorem 2 that $\mathcal{F}$ is the minimal list of forbidden minors for ORTs.

The graph obtained from a complete bipartite graph $K_{1,n}$ by replacing each edge with a path of length 3 is called an $(n, 3)$-*spider*. The vertex with degree n in an $(n, 3)$-spider T is called the *center* of T. Notice that a $(3, 3)$-spider is a 3-claw, and a $(5, 3)$-spider is F_0 in Fig. 5.

In this section, we denote a path P of length k as a sequence of vertices $(v_0, v_1, \ldots, v_k)$ on P. We say that P is a path from v_0 to v_k or a (v_0, v_k)-path. Two vertices v_0 and v_k are called the *end-vertices* of P, and the remaining vertices are called the *internal vertices* of P. Notice that in a tree, any two vertices are connected by the unique path (see [2] for example). We define the following (see Fig. 6).

- A tree is said to be N_Y-*type* if it is obtained from two paths of arbitrary lengths by identifying an end-vertex of a path with an internal vertex of the other path. All the vertices are colored *black*.
- A tree is said to be N_0-*type* if it is obtained from two paths of arbitrary lengths by identifying an internal vertex of a path with an internal vertex of the other path. All the vertices are colored *black*.
- A tree is said to be N_i-*type*, $1 \le i \le 4$, if it is obtained from a path P of length i together with two paths P_1 and P_2 of arbitrary lengths by identifying one end-vertex of P with an internal vertex of P_1 and identifying the other end-vertex of P with an internal vertex of P_2. The vertices on P_1 and P_2 are colored *black*, and the vertices on P, except for the end-vertices, are colored *white*.

A tree T is called an N_i-*tree*, $i \in \{Y, 0, 1, \ldots, 4\}$, if T contains an N_i-type tree T_i as a subtree such that every vertex of T is within distance 2 from a black vertex of T_i. Now, we have the following.

Theorem 2. *The following statements are equivalent for a tree T:*

(i) *T is an ORT;*
(ii) *T contains no A5Es;*
(iii) *T contains no tree in $\mathcal{F}$ as a minor;*

(iv) *T is a 2-DORT or N_i-tree for some* $i \in \{Y, 0, 1, \dots, 4\}$*;*
(v) *T can be split into two 2-DORTs.*

Proof. (i) ⟹ (ii): It is immediate from Theorem E.

(ii) ⟹ (iii): Any tree in $\mathcal{F}$ has an A5E as shown in Fig. 5. Thus, if T contains a tree in $\mathcal{F}$ as a minor, then T contains an A5E by the definition of A5Es.

(iii) ⟹ (iv): It suffices to show that if a tree is neither a 2-DORT nor an N_i-tree for any $i \in \{Y, 0, 1, \dots, 4\}$, it contains a tree in $\mathcal{F}$ as a minor. Let T be such a tree. It follows from Theorem B that T contains at least one $(3, 3)$-spider as a subtree, for otherwise T is a 2-DORT. We distinguish three cases.

Case 1 T contains an $(n, 3)$-spider for some $n \geq 5$ as a subtree: Since a $(5, 3)$-spider is F_0, T contains F_0 as a minor.

Case 2 T contains a $(4, 3)$-spider as a subtree but contains no $(n, 3)$-spiders for any $n \geq 5$: Let c_1 be the center of the $(4, 3)$-spider. T must contain another $(3, 3)$-spider whose center is not c_1, for otherwise T is an N_0-tree. Let c_2 be the center of another $(3, 3)$-spider in T. Let T' be the tree obtained from T by contracting the edges on the (c_1, c_2)-path. Since T' contains F_0 as a subtree, T contains F_0 as a minor.

Case 3 T contains a $(3, 3)$-spider as a subtree but contains no $(n, 3)$-spiders for any $n \geq 4$: Suppose T contains at least three $(3, 3)$-spiders with distinct centers. Let T' be the tree obtained from T by contracting the edges on the path from one of the centers to another center. Since T' contains the $(4, 3)$-spider and the $(3, 3)$-spider with distinct centers, T' contains F_0 as a minor as shown in **Case 2**. Hence, T contains F_0 as a minor.

Now, we assume that T contains at most two $(3, 3)$-spiders with distinct centers. Then, T must contain two $(3, 3)$-spiders, for otherwise T is a 2-DORG or an N_Y-tree. Let c_1 and c_2 be the centers of the $(3, 3)$-spiders, let P be the (c_1, c_2)-path, and let $\mathbf{dist}(c_1, c_2)$ be the length of P. We further distinguish five cases.

Case 3-1 $\mathbf{dist}(c_1, c_2) = 1$ (see Fig. 6(c)): T is an N_1-tree, a contradiction.

Case 3-2 $\mathbf{dist}(c_1, c_2) = 2$ (see Fig. 6(d)): Let $P = (c_1, u_1, c_2)$. T must have two vertices v_1 and v_2 together with two edges (u_1, v_1) and (v_1, v_2), for otherwise T is an N_2-tree. Then, T contains F_1 as a minor.

Case 3-3 $\mathbf{dist}(c_1, c_2) = 3$ (see Fig. 6(e)): Let $P = (c_1, u_1, u_2, c_2)$. T must have two vertices v_1 and v_2 together with two edges (u_i, v_1) and (v_1, v_2), $i \in \{1, 2\}$, for otherwise T is an N_3-tree. Then, T contains F_1 as a minor.

Case 3-4 $\mathbf{dist}(c_1, c_2) = 4$ (see Fig. 6(f)): Let $P = (c_1, u_1, u_2, u_3, c_2)$. T must have a vertex v_1 together with an edge (u_2, v_1), or two vertices v_1 and v_2 together with two edges (u_i, v_1) and (v_1, v_2), $i \in \{1, 3\}$, for otherwise T is an N_4-tree. If T has a vertex v_1 together with an edge (u_2, v_1), T contains F_2 as a minor. If T has two vertices v_1 and v_2 together with two edges (u_i, v_1) and (v_1, v_2), $i \in \{1, 3\}$, T contains F_1 as a minor.

Case 3-5 $\mathbf{dist}(c_1, c_2) \geq 5$: T contains F_3 as a minor.

(iv) ⟹ (v): We can see from Theorem B that if we split an N_i-tree ($i \in \{Y, 0, 1, 2\}$) at c_1, then we obtain two 2-DORTs. Similarly, if we split an N_i-tree ($i \in \{3, 4\}$) at u_2, then we obtain two 2-DORTs.

(v) ⟹ (i): Suppose T can be split into two 2-DORTs, that is, T can be obtained from two 2-DORTs T_1 and T_2 by identifying a vertex $u \in V(T_1)$ with $v \in V(T_2)$. Let P_i be the spine (longest path) of T_i for each $i \in \{1, 2\}$. Let $V_1(T_i)$ be the set of vertices with distance 1 from the vertices of P_i for each $i \in \{1, 2\}$, and let $V_2(T_i)$ be the set of vertices with distance 2 from the vertices of P_i. Let $P_1 = (u_0, u_1, \dots, u_p)$, and $P_2 = (v_0, v_1, \dots, v_q)$. We distinguish six cases and construct an orthogonal ray representation of T for each of them.

Case 1 $u \in V(P_1)$ and $v \in V(P_2)$: Let $u = u_i \in V(P_1)$ and $v = v_j \in V(P_2)$. For the vertices $u_{i+k} \in V(P_1)$, $-i \leq k \leq p - i$, define the corresponding rays as follows (see Fig. 7):

- let R_{u_i} be the rightward ray with endpoint $(-2, 0)$;
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k, k-1)$ if k is positive and odd;
- let $R_{u_{i+k}}$ be the rightward ray with endpoint $(k-1, k)$ if k is positive and even;
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k-1, -k-1)$ if k is negative and odd; and
- let $R_{u_{i+k}}$ be the leftward ray with endpoint $(k, -k)$ if k is negative and even.

For the vertices $v_{j+k} \in V(P_2)$, $-j \leq k \leq q - j$, define the corresponding rays as follows:

- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k+1, -k+1)$ if k is positive and odd;
- let $R_{v_{j+k}}$ be the rightward ray with endpoint $(k, -k)$ if k is positive and even;
- let $R_{v_{j-1}}$ be the downward ray with endpoint $(0, 0)$;
- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k+1, k)$ if $k \leq -3$ and odd; and
- let $R_{v_{j+k}}$ be the leftward ray with endpoint $(k+2, k-1)$ if k is negative and even.

The rays corresponding to the remaining vertices of T can be added to the specified regions shown in Fig. 7.

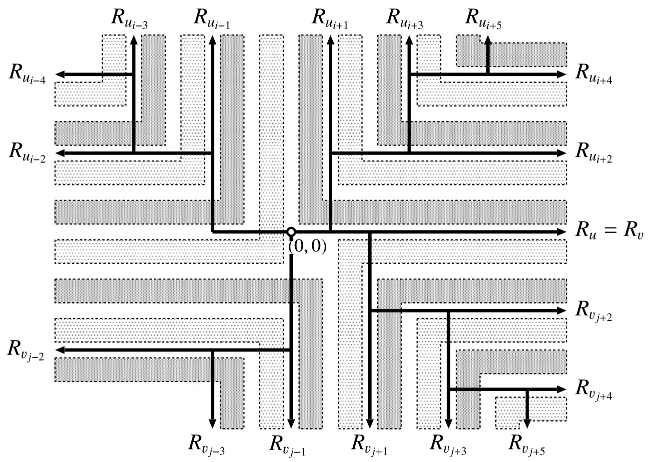


Fig. 7. An orthogonal ray representation of T when $u \in V(P_1)$ and $v \in V(P_2)$ (Case 1).

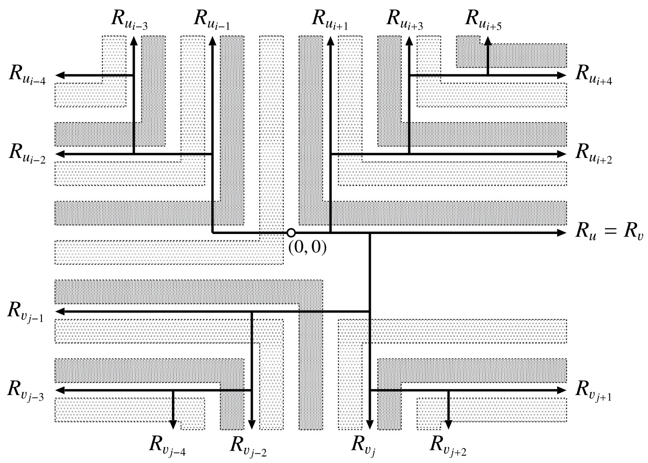


Fig. 8. An orthogonal ray representation of T when $u \in V(P_1)$ and $v \in V_1(T_2)$ (Case 2).

Case 2 $u \in V(P_1)$ and $v \in V_1(T_2)$: Let $u = u_i \in V(P_1)$, and let $v_j \in V(P_2)$ be the vertex on P_2 adjacent to v. For the vertices of $V(P_1)$, define the corresponding rays as shown in **Case 1**. For the vertices $v_{j+k} \in V(P_2)$, $-j \le k \le q - j$, define the corresponding rays as follows (see Fig. 8):

- let R_{v_j} be the downward ray with endpoint $(2, 0)$;
- let $R_{v_{j+k}}$ be the rightward ray with endpoint $(k + 1, -k - 3)$ if k is positive and odd;
- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k + 2, -k - 2)$ if k is positive and even;
- let $R_{v_{j-1}}$ be the leftward ray with endpoint $(2, -2)$;
- let $R_{v_{j+k}}$ be the leftward ray with endpoint $(k + 2, k - 1)$ if $k \le -3$ and odd; and
- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k + 1, k)$ if k is negative and even.

The rays corresponding to the remaining vertices of T can be added to the specified regions shown in Fig. 8.

Case 3 $u \in V(P_1)$ and $v \in V_2(T_2)$: Let $u = u_i \in V(P_1)$, let $v' \in V_1(T_2)$ be the vertex of $V_1(T_2)$ adjacent to v, and let $v_j \in V(P_2)$ be the vertex on P_2 adjacent to v'. For the vertices of $V(P_1)$, define the corresponding rays as shown in **Case 1**. For the vertex v' and the vertices $v_{j+k} \in V(P_2)$, $-j \le k \le q - j$, define the corresponding rays as follows (see Fig. 9):

- let $R_{v'}$ be the downward ray with endpoint $(2, 0)$;
- let R_{v_j} be the leftward ray with endpoint $(4, -2)$;
- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k + 3, -k - 1)$ if k is positive and odd;
- let $R_{v_{j+k}}$ be the rightward ray with endpoint $(k + 2, -k - 2)$ if k is positive and even;
- let $R_{v_{j+k}}$ be the downward ray with endpoint $(k + 1, k - 1)$ if k is negative and odd; and
- let $R_{v_{j+k}}$ be the leftward ray with endpoint $(k + 2, k - 2)$ if k is negative and even.

The rays corresponding to the remaining vertices of T can be added to the specified regions shown in Fig. 9.

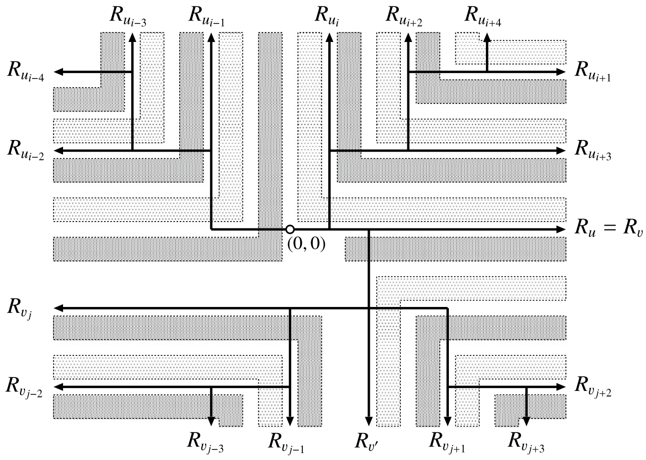


Fig. 9. An orthogonal ray representation of T when $u \in V(P_1)$ and $v \in V_2(T_2)$ (Case 3).

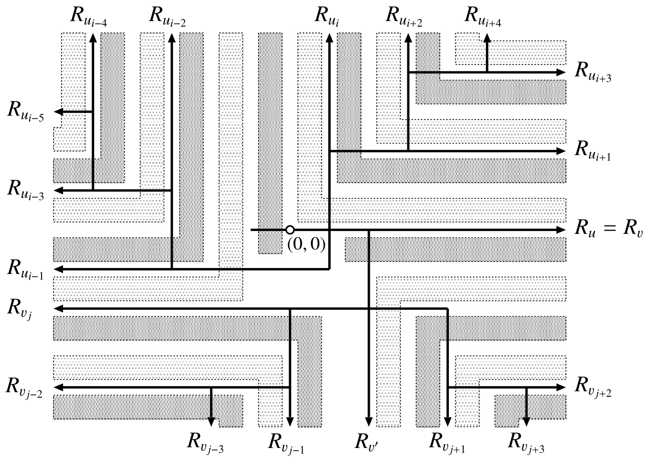


Fig. 10. An orthogonal ray representation of T when $u \in V_1(T_1)$ and $v \in V_2(T_2)$ (Case 5).

Case 4 $u \in V_1(T_1)$ and $v \in V_1(T_2)$: Let u_i be the vertex on P_1 adjacent to u. Since T can be split at u_i into two 2-DORTs (see Fig. 6(d)), T has an orthogonal ray representation as shown in **Case 3**.

Case 5 $u \in V_1(T_1)$ and $v \in V_2(T_2)$: Let $u_i \in V(P_1)$ be the vertex on P_1 adjacent to u, let $v' \in V_1(T_2)$ be the vertex of $V_1(T_2)$ adjacent to v, and let $v_j \in V(P_2)$ be the vertex on P_2 adjacent to v'. For the vertex v' and the vertices of $V(P_2)$, define the corresponding rays as shown in **Case 3**. For the vertex u and the vertices $u_{i+k} \in V(P_1)$, $-i \le k \le p-i$, define the corresponding rays as follows (see Fig. 10):

- let R_u be the rightward ray with endpoint $(-1, 0)$;
- let R_{u_i} be the upward ray with endpoint $(1, -1)$;
- let $R_{u_{i+k}}$ be the rightward ray with endpoint $(k, k+1)$ if k is positive and odd;
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k+1, k)$ if k is positive and even;
- let $R_{u_{i-1}}$ be the leftward ray with endpoint $(1, -1)$;
- let $R_{u_{i+k}}$ be the leftward ray with endpoint $(k, -k-2)$ if $k \le -3$ and odd; and
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k-1, -k-3)$ if k is negative and even.

The rays corresponding to the remaining vertices of T can be added to the specified regions shown in Fig. 10.

Case 6 $u \in V_2(T_1)$ and $v \in V_2(T_2)$: Let $u' \in V_1(T_1)$ be the vertex of $V_1(T_1)$ adjacent to u, and let $u_i \in V(P_1)$ be the vertex on P_1 adjacent to u'. Let $v' \in V_1(T_2)$ be the vertex of $V_1(T_2)$ adjacent to v, and let $v_j \in V(P_2)$ be the vertex on P_2 adjacent to v'. For the vertex v' and the vertices of $V(P_2)$, define the corresponding rays as shown in **Case 3**. For the vertices u, u', and $u_{i+k} \in V(P_1)$, $-i \le k \le p-i$, define the corresponding rays as follows (see Fig. 11):

- let R_u be the rightward ray with endpoint $(1, 0)$;
- let $R_{u'}$ be the upward ray with endpoint $(1, 0)$;

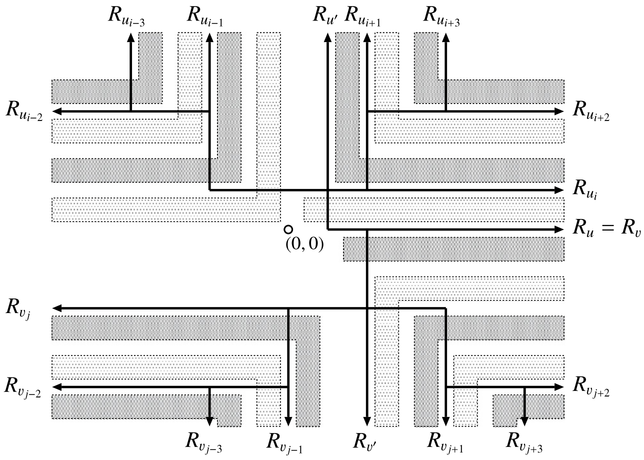


Fig. 11. An orthogonal ray representation of T when $u \in V_2(T_1)$ and $v \in V_2(T_2)$ (Case 6).

Input:	A tree T.
Output:	YES, if T is an ORT. NO, otherwise.
Step 1:	Set T' to be the tree obtained from T by deleting all leaves in T. Set T'' to be the tree obtained from T' by deleting all leaves in T'.
Step 2:	If T'' has a vertex of degree at least 5, output NO and halt.
Step 3:	If T'' has a vertex of degree 4 and another vertex of degree at least 3, output NO and halt.
Step 4:	If T'' has a vertex of degree 4, output YES and halt.
Step 5:	If T'' has at least three vertices of degree 3, output NO and halt.
Step 6:	If T'' has at most one vertex of degree 3, output YES and halt.
Step 7:	Set c_1 and c_2 to be the two vertices of degree 3 in T''. Set P to be the path from c_1 to c_2. If the length of P is greater than 4, output NO and halt.
Step 8:	Assign color black to the vertices of T'' including c_1 and c_2, and excluding other vertices on P. If every vertex of T is within distance 2 from a black vertex, output YES, and halt. Else, output NO and halt.

Fig. 12. Algorithm 1.

- Let R_{u_i} be the rightward ray with endpoint $(-2, 1)$;
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k + 1, k)$ if k is positive and odd;
- let $R_{u_{i+k}}$ be the rightward ray with endpoint $(k, k + 1)$ if k is positive and even;
- let $R_{u_{i+k}}$ be the upward ray with endpoint $(k - 1, -k)$ if k is negative and odd; and
- let $R_{u_{i+k}}$ be the leftward ray with endpoint $(k, -k + 1)$ if k is negative and even.

The rays corresponding to the remaining vertices of T can be added to the specified regions shown in Fig. 11. □

4.2. A linear-time recognition algorithm

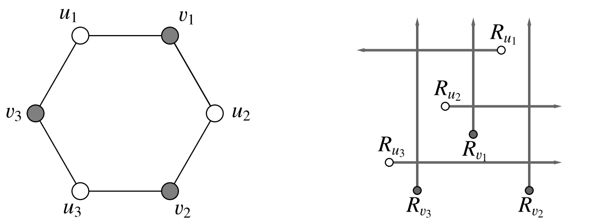
We show in Fig. 12 a linear-time algorithm to recognize ORTs. Our algorithm is based on Theorem 2 (iv). Since all the steps can be done in linear time, we have the following.

Theorem 3. *Algorithm* 1 *solves the recognition problem for ORTs in linear time.* □

5. Three-directional orthogonal ray graphs

We first show a necessary condition for 3-DORGs. To prove the condition, we use the following.

Lemma 4. *For any permutation π on $\{0, 1, \ldots, 2k\}$, $k \geq 1$, there exists an integer i, $0 \leq i \leq 2k$, such that $\pi_i < \pi_{i+k+1} < \pi_{i+1}$ or $\pi_i > \pi_{i+k+1} > \pi_{i+1}$ (subscripts are modulo $2k + 1$).*



(a) A cycle C_6. (b) The representation of C_6.

Fig. 13. A cycle C_6 and its orthogonal ray representation.

Proof. It suffices to show that there exists an integer i, $0 \le i \le 2k$, such that $(\pi_i - \pi_{i+k+1})(\pi_{i+k+1} - \pi_{i+1}) > 0$. Suppose contrary that $(\pi_i - \pi_{i+k+1})(\pi_{i+k+1} - \pi_{i+1}) < 0$ for any i, $0 \le i \le 2k$. Then, we have that

$$0 > \prod_{i=0}^{2k}(\pi_i - \pi_{i+k+1})(\pi_{i+k+1} - \pi_{i+1}) = \prod_{i=0}^{2k}(\pi_i - \pi_{i+k+1})^2 > 0,$$

a contradiction. Thus, we have the lemma. □

The following shows the necessary condition for 3-DORGs.

Theorem 5. *Any 3-DORG contains no edge-asteroids.*

Proof. Let G be a 3-DORG with bipartition (U, V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$. We assume without loss of generality that every R_v, $v \in V$, is an upward ray. Let (x_w, y_w) be the endpoint of $R_w \in \mathcal{R}(G)$, $w \in V(G)$. Since the vertical rays in $\mathcal{R}(G)$ are disjoint, every x_v, $v \in V$, is distinct.

We prove the theorem by contradiction. Suppose that G has an edge-asteroid $\{e_0, e_1, \dots, e_{2k}\}$ of size $2k+1$, $k \ge 1$, where $e_i = (u_i, v_i)$ with $u_i \in U$ and $v_i \in V$ for any i, $0 \le i \le 2k$. Notice that two edges in an edge-asteroid may share a common vertex as shown in Fig. 1(a). Thus, it is possible that $e_i \ne e_j$ but $v_i = v_j$. However, we can see that for any i, e_i and e_{i+k+1} share no common vertex and have no edge joining them, for otherwise it contradicts the definition of edge-asteroids. Similarly, e_{i+1} and e_{i+k+1} share no common vertex and have no edge joining them.

Now, we consider the upward rays R_{v_i}, $0 \le i \le 2k$, corresponding to the end-vertices of the edges in the edge-asteroid. Let π_i be the position of R_{v_i} among the upward rays numbered from left to right, that is, we define the permutation π on $\{0, 1, \dots, 2k\}$ such that $\pi_i < \pi_j$ if and only if $x_{v_i} \le x_{v_j}$ for any i and j (Tie breaking arbitrarily). We have from Lemma 4 that there exists an integer i such that $\pi_i < \pi_{i+k+1} < \pi_{i+1}$ or $\pi_i > \pi_{i+k+1} > \pi_{i+1}$ (subscripts are modulo $2k+1$), that means the upward ray $R_{v_{i+k+1}}$ is in between R_{v_i} and $R_{v_{i+1}}$. Two rays $R_{u_{i+k+1}}$ and $R_{v_{i+k+1}}$ divide the plane into two regions, and R_{v_i} and $R_{v_{i+1}}$ are in the different regions since R_{v_i} and $R_{v_{i+1}}$ are upward rays. Hence, any path from e_i to e_{i+1} must have a vertex adjacent to u_{i+k+1} or v_{i+k+1}, contradicting the definition of edge-asteroids. Thus, G contains no edge-asteroids. □

We have the following from Theorem 5.

Corollary 6. *A cycle C_{2n} of length $2n$ is a 3-DORG if and only if $2 \le n \le 3$.*

Proof. We can see that C_{2n} is a 3-DORG if $2 \le n \le 3$ (see Fig. 13 for a cycle of length 6). A cycle of length 8 has an edge-asteroid as shown in Fig. 1(a). It follows that any cycle of length at least 8 has an edge-asteroid, and it is not a 3-DORG by Theorem 5. Thus, we have the corollary. □

In Section 1, we include the hierarchy of graph classes related to ORGs [25], but it does not mention anything about 3-DORGs. Now, we have the following.

Corollary 7. *{2-Directional Orthogonal Ray Graphs} ⊂ {3-Directional Orthogonal Ray Graphs} ⊂ {Orthogonal Ray Graphs}, where $X \subset Y$ indicates a set X is a proper subset of Y.*

Proof. Since the inclusions are trivial by the definitions, we show some separating examples for these classes. A cycle of length 6 is a 3-DORG by Corollary 6, but not a 2-DORG by Theorem A. Thus, the class of 2-DORGs is a proper subset of the class of 3-DORGs. Also, cycles of length 8, 10, and 12 are ORGs by Theorem C, but not 3-DORGs by Corollary 6. Thus, the class of 3-DORGs is a proper subset of the class of ORGs, and we have the corollary. □

It is also shown in [25] that the class of 2-DORGs is a proper subset of the class of chordal bipartite graphs, but the classes of ORGs and chordal bipartite graphs are incomparable. We have the following.

Corollary 8. *The classes of 3-DORGs and chordal bipartite graphs are incomparable.*

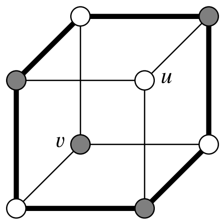
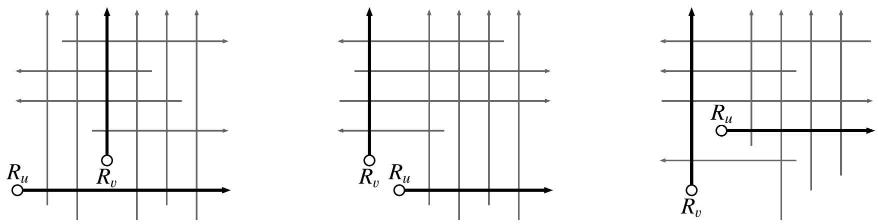


Fig. 14. The 3-dimensional hypercube Q_3. Bold edges denote a cycle of length 6.



(a) Case 1: $x_u < x_v$ and $y_u < y_v$. (b) Case 2: $x_v < x_u$ and $y_u < y_v$. (c) Case 3: $x_v < x_u$ and $y_v < y_u$.

Fig. 15. The three cases of the relative position of the rays corresponding to u and v.

Proof. The class of 3-DORGs is not a subset of chordal bipartite graphs, since a cycle of length 6 is a 3-DORG by Corollary 6, while it is not a chordal bipartite graph by the definition. On the other hand, chordal bipartite graphs having edge-asteroids are shown in [25]. Since these graphs are not 3-DORGs by Theorem 5, the class of chordal bipartite graphs is not a subset of the class of 3-DORGs. Thus, we have the corollary. □

Moreover, the following can be obtained from Theorems 5 and A.

Corollary 9. *A 3-DORG is a 2-DORG if and only if it is a chordal bipartite graph.* □

We also have the following from Corollary 9, since any tree is a chordal bipartite graph.

Corollary 10. *A tree is a 3-DORT if and only if it is a 2-DORT.* □

Note that Corollary 10 implies a linear-time recognition algorithm for 3-DORTs as shown in Section 2.

In the rest of this section, we show that the 3-dimensional hypercube Q_3 (see Fig. 14) is not a 3-DORGs, while it contains no edge-asteroids. It follows that the necessary condition in Theorem 5 is not sufficient to characterize 3-DORGs. The characterization of 3-DORGs remains an open question. We first show the following.

Lemma 11. *If G is a 3-DORG with bipartition (U, V) and two vertices $u \in U$ and $v \in V$ are not adjacent, then the subgraph of G induced by $N(u) \cup N(v)$ is a 2-DORG.*

Proof. Let $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$ be an orthogonal ray representation of G. We assume without loss of generality that every $R_{v'}$, $v' \in V$, is an upward ray. We further assume that R_u is a rightward ray. Let G_{uv} be the subgraph of G induced by $N(u) \cup N(v)$, and let $\mathcal{R}(G_{uv})$ be the restriction of $\mathcal{R}(G)$ to the rays corresponding to the vertices of G_{uv}. Let (x_w, y_w) be the endpoint of $R_w \in \mathcal{R}(G)$, $w \in V(G)$. We distinguish three cases according to the relative position of the endpoints of R_u and R_v.

Case 1 $x_u < x_v$ and $y_u < y_v$ (see Fig. 15(a)): A bipartite graph G' with bipartition (U', V') is called a *convex graph* [10] if there exists a total ordering of the vertices in V' such that for any $u' \in U'$, the vertices in $N(u')$ occur consecutively in the ordering. It is known that any convex graph is a 2-DORG [25]. We show in this case that G_{uv} is a convex graph. Let $(v_1, v_2, \ldots, v_p)$ be the total ordering of $N(u)$ such that for any v_i and v_j, $i < j$ if and only if $x_{v_i} < x_{v_j}$. Since $y_u < y_v$, we can see that for any $u' \in N(v)$, $N(u')$ occur consecutively in such ordering. Thus, G_{uv} is a convex graph.

Case 2 $x_v < x_u$ and $y_u < y_v$ (see Fig. 15(b)): We have $x_v < x_u < x_{v'}$ for every $v' \in N(u)$. Thus, we obtain the orthogonal ray representation of G_{uv} from $\mathcal{R}(G_{uv})$ by replacing each rightward ray with a leftward ray having the same y-coordinate whose endpoint is on the right of all the vertical rays intersecting R_u. It follows that G_{uv} is a 2-DORG.

Case 3 $x_v < x_u$ and $y_v < y_u$ (see Fig. 15(c)): The proof is similar to that in **Case 2**, and is omitted. □

Let G be a bipartite graph with bipartition (U, V) that is not a 2-DORG, and let G' be the graph obtained from G by adding two non-adjacent vertices u and v to G such that u is adjacent to every vertex in V and v is adjacent to every vertex in U. In other words, $V(G') = V(G) \cup \{u, v\}$ and $E(G') = E(G) \cup \{(u, z) \mid z \in V\} \cup \{(w, v) \mid w \in U\}$. Then, G' is not a 3-DORG by Lemma 11. Now, we have the following.

Theorem 12. *The 3-dimensional hypercube Q_3 is not a 3-DORG, while it contains no edge-asteroids.*

Proof. Notice that Q_3 is obtained from C_6 (the cycle of length 6) together with two non-adjacent vertices u and v by joining u to the vertices in one color class of C_6 and joining v to the vertices in the other color class of C_6 (see Fig. 14). Since C_6 is not a 2-DORG by Theorem A, Q_3 is not a 3-DORG.

It remains to show that Q_3 contains no edge-asteroids. A set of edges $\{e_0, e_1, \ldots, e_{2k}\} \subseteq E(G)$, $k \geq 1$, of a graph G is an edge-asteroid if and only if for any i, $0 \leq i \leq 2k$, two edges e_i and e_{i+1} are in the same connected component of the subgraph of G obtained by deleting all the vertices adjacent to the end-vertices of e_{i+k+1} (subscripts are modulo $2k + 1$). We can observe that for any edge e of Q_3, only one edge remains after deleting all the vertices adjacent to the end-vertices of e. Thus, Q_3 contains no edge-asteroids, and we have the theorem. □

Acknowledgments

The authors are grateful to Dr. Anish Man Singh Shrestha for stimulating discussion. The third author was partially supported by JSPS Grant-in-Aid for JSPS Fellows (26·8924). The last author was partially supported by JSPS KAKENHI Grant Number 26330007.

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