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PAPER

Assignment of Intervals to Parallel Tracks with Minimum Total Cross-Talk

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SUMMARY The most basic cross-talk minimization problem is to assign given n intervals to n parallel tracks where the cross-talk is defined between two intervals assigned to the adjacent tracks with the amount linear to parallel running length. This paper solves the problem for the case when any pair of intervals intersects and the objective is to minimize the sum of cross-talks. We begin the discussion with the fact that twice the sum of lengths of $\lfloor n/2 \rfloor$ shortest intervals is a lower bound. Then an interval set that attains this lower bound is characterized with a simple assignment algorithm. Some additional considerations provide the minimum cross-talk for the other interval sets. The main procedure is to sort the intervals twice with respect to the length of left and right halves of intervals.

key words: *minimum cross-talk, assignment, intersecting interval sets*

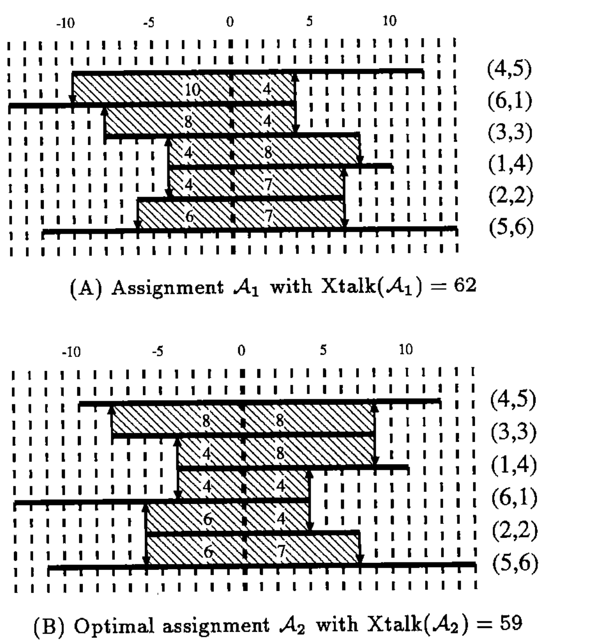
1. Introduction

Along with the development of high-density and high-speed circuits, the layout design is confronting a new difficulty in routing, i.e. the minimization of the cross-talk between signals on wires. Various cross-talk models have been proposed according to the devices, design styles, and specifications. However, there have been few contributions that give any provable optimum. Most of contributions, as far as the authors know ([1]–[5]), are heuristics algorithms. To our knowledge, only one preceding result is found in Zhou and Wong [6] where an optimal river routing problem is solved on the same model as is adopted in this paper. A motivation of this paper is to present any non-trivial fact which could be a principle or a rule-of-thumb in routing.

This paper assumes the simplest model called the *adjacent-cross-talk model*: the cross-talk is observed only between two adjacent parallel running intervals and the quantity is proportional to the parallel running length. The problem is to assign given n intervals to n parallel tracks such that the sum of cross-talks is minimum. This paper solves the problem for the *intersecting interval set* which is a set of intervals on the real line such that any two intervals have non-empty cross-talk when they are adjacent. In Fig. 1, an intersecting interval set $\mathcal{S}$ and two assignments are shown. Each figure between intervals represents the amount of cross-talk. The total cross-talk of assignment $\mathcal{A}_1$ is 62 and that of assignment A_2 is 59. ($\mathcal{A}_2$ is optimal as will be proved in Sect. 5.)

Our problem has been graph theoretically formulated as follows: The cross-talk is represented by an edge-weighted complete graph where a vertex corresponds to an interval and the weight of an edge is the cross-talk between two intervals corresponding to its end vertices when the intervals are adjacent. The optimal interval assignment problem is equivalent to find a minimum weight Hamiltonian path of the graph since the total cross-talk of an assignment of n intervals to the n tracks corresponds to the length of a Hamiltonian path.

Finding a minimum weight Hamiltonian path of a general graph is $\mathcal{NP}$-hard and trivially harder than our problem. The converse had not been proved but still this similarity has been making us doubt if our prob-



(A) Assignment $\mathcal{A}_1$ with Xtalk($\mathcal{A}_1$) = 62

(B) Optimal assignment $\mathcal{A}_2$ with Xtalk($\mathcal{A}_2$) = 59

Fig. 1 Input $\mathcal{S}$ = {[−4, 10], [−6, 7], [−8, 8], [−10, 12], [−12, 14], [−14, 4]}, Two assignments A_1, A_2, cross-talks, and double-orders.

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lem is $\mathcal{NP}$-hard. This paper clears that the problem in our case so restricted that the problem is $\mathcal{P}$. It will also reveal that the problem is easier to treat directly than using graph theoretical formulation. Or more precisely, a polynomial time algorithm is given to construct an optimal assignment. The algorithm is simply to sort the intervals in two ways, thus, the time complexity of the algorithm is $O(n \log n)$.

The rest of the paper is organized as follows. Section 2 is for definitions. In Sect. 3, a lower bound is presented. In Sect. 4, a notion of perfect assignment is introduced. In Sect. 5, the classes of interval sets for which no perfect assignment is possible are considered. Section 6 is the conclusion.

2. Preliminaries

The input of the problem is the set $\mathcal{S} = \{I_1, \ldots, I_n\}$ of *intervals* on the real line, x-axis, where each interval lying between $x = a$ and $x = b$ $(a < b)$ is represented as $I = [a, b]$. The length is $|I| = b - a$.

For two intervals $I_i = [a_i, b_i]$ and $I_j = [a_j, b_j]$ of $\mathcal{S}$,

$$I_i \wedge I_j = \begin{cases} [\max\{a_i, a_j\}, \min\{b_i, b_j\}] \\ \quad\quad \text{if} \quad \max\{a_i, a_j\} \leqq \min\{b_i, b_j\} \\ \emptyset \quad\quad \text{otherwise.} \end{cases}$$

The interval set $\mathcal{S}$ considered in this paper is assumed to satisfy.

Intersecting Assumption: Any two intervals are intersecting, i.e. $I_i \wedge I_j \neq \emptyset$ for any $I_i, I_j \in \mathcal{S}$.

Then, without loss of generality, we assume that every interval contains $x = 0$. By this assumption, each interval I is represented by a pair of divided halves $I^- = [a, 0]$ and $I^+ = [0, b]$ $(a \leqq 0 \leqq b)$, which are called the *negative* and *positive parts* of I, respectively.

The intervals are embedded into the regularly spaced n horizontal tracks in the channel. The i-th track from the top in the channel is labeled i, for simplicity. An *assignment* $\mathcal{A}$ assigns the intervals to the tracks, one interval to one track. Thus, an *assignment* $\mathcal{A}$ of $\mathcal{S}$ is an integer valued one-to-one-function

$$\mathcal{A} : \mathcal{S} \rightarrow \{1, 2, \cdots, n\},$$

representing that $\mathcal{A}$ assigns $I \in \mathcal{S}$ to the $\mathcal{A}(I)$-th track. Two intervals I_i and I_j are said to be *adjacent* in $\mathcal{A}$ if they are assigned to the adjacent tracks in $\mathcal{A}$, i.e. $|\mathcal{A}(I_i) - \mathcal{A}(I_j)| = 1$.

An assignment $\mathcal{A}$ is evaluated by the smallness of the total cross-talk

$$\mathrm{Xtalk}(\mathcal{A}) = \sum_{I_i, I_j:\ \text{adjacent in } \mathcal{A}} |I_i \wedge I_j|.$$

We consider the total cross-talks in the positive part and negative part separately as

$$\mathrm{Xtalk}(\mathcal{A}) = \mathrm{Xtalk}^-(\mathcal{A}) + \mathrm{Xtalk}^+(\mathcal{A})$$

where

$$\mathrm{Xtalk}^-(\mathcal{A}) = \sum_{I_i, I_j:\ \text{adjacent in } \mathcal{A}} |I_i^- \wedge I_j^-|,$$

$$\mathrm{Xtalk}^+(\mathcal{A}) = \sum_{I_i, I_j:\ \text{adjacent in } \mathcal{A}} |I_i^+ \wedge I_j^+|.$$

If $n \leqq 2$, the total cross-talk of any assignment is unique. In the following, we assume that $n \geqq 3$.

A sorting function $O^+(I)$ is defined for $I \in \mathcal{S}$ with respect to the shortness of the length in the positive part of I, i.e. $O^+(I) = k$ if I^+ is k-th shortest in $\{I_1^+, I_2^+, \ldots, I_n^+\}$. Similarly, $O^-(I)$ is defined with respect to the shortness of the length in the negative part. Thus, an interval I has the *double-order* $(O^-(I), O^+(I))$. In Fig. 1, the double-order is shown in the right of each interval.

An interval I such that $O^+(I) = k$ is denoted as $I_{k/+}$, reading the k-th shortest interval in the positive part. $I_{k/-}$ is defined similarly. Define four subsets of $\mathcal{S}$ as follows.

$$\mathcal{S}_{s/-} = \{I_{k/-} | 1 \leqq k < n/2\},$$
$$\mathcal{S}_{l/-} = \{I_{k/-} | n/2 < k \leqq n\},$$
$$\mathcal{S}_{s/+} = \{I_{k/+} | 1 \leqq k < n/2\},$$
$$\mathcal{S}_{l/+} = \{I_{k/+} | n/2 < k \leqq n\}.$$

Subscripts "$s/\pm$" and "$l/\pm$" stand for "shorter intervals in the positive or negative part" and "longer intervals in the positive or negative part," respectively. Note that $|\mathcal{S}_{s/-}| = |\mathcal{S}_{s/+}| = \lceil \frac{n}{2} \rceil - 1$ and $|\mathcal{S}_{l/-}| = |\mathcal{S}_{l/+}| = \lceil \frac{n}{2} \rceil$. If n is odd, $\mathcal{S}_{s/-} \cup \mathcal{S}_{l/-} = \mathcal{S}_{s/+} \cup \mathcal{S}_{l/+} = \mathcal{S}$. Otherwise, $I_{(n/2)/+}$ and $I_{(n/2)/-}$ are missing in these sets.

Further subsets are defined.

$$\mathcal{S}_{s,s} = \mathcal{S}_{s/-} \cap \mathcal{S}_{s/+},$$
$$\mathcal{S}_{s,l} = \mathcal{S}_{s/-} \cap \mathcal{S}_{l/+},$$
$$\mathcal{S}_{l,s} = \mathcal{S}_{l/-} \cap \mathcal{S}_{s/+},$$
$$\mathcal{S}_{l,l} = \mathcal{S}_{l/-} \cap \mathcal{S}_{l/+}.$$

The intervals which are $\lfloor n/2 \rfloor$-th shortest in either part will play key roles and be frequently referred to. In the following, we let $\alpha = \lfloor n/2 \rfloor$.

A brief consideration leads to the following fact.

Lemma 1:

$$|\mathcal{S}_{l,l}| - |\mathcal{S}_{s,s}| = 1 + \delta_1,$$
$$|\mathcal{S}_{s,l}| - |\mathcal{S}_{l,s}| = \delta_2$$

where (δ_1, δ_2) is:

Case 1. if n is odd, $(0, 0)$;

otherwise (n is even),

Case 2-a. if $I_{\alpha/+} = I_{\alpha/-}$, $(0, 0)$;

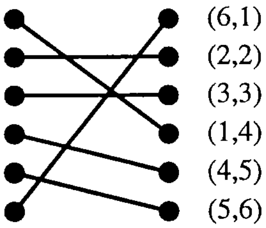

Fig. 2 The bipartite representation of $\mathcal{S}$ given in Fig. 1.

Case 2-b. if $I_{\alpha/+} \in \mathcal{S}_{s/-}$ and $I_{\alpha/-} \in \mathcal{S}_{s/+}$, $(1, 0)$;

Case 2-c. if $I_{\alpha/+} \in \mathcal{S}_{s/-}$ and $I_{\alpha/-} \in \mathcal{S}_{l/+}$, $(0, -1)$;

Case 2-d. if $I_{\alpha/+} \in \mathcal{S}_{l/-}$ and $I_{\alpha/-} \in \mathcal{S}_{s/+}$, $(0, 1)$;

Case 2-e. if $I_{\alpha/+} \in \mathcal{S}_{l/-}$ and $I_{\alpha/-} \in \mathcal{S}_{l/+}$, $(-1, 0)$. □

Note that cases (1) and (2-a)∼(2-e) are exclusive and form a classification of all possible interval sets. These classes are easily understood by the bipartite representation defined as follows. In the left and right columns, lay n vertices. An interval I is represented by an edge connecting the $O^-(I)$-th vertex in the left and $O^+(I)$-th vertex in the right. As an example, the bipartite representation of the interval set given in Fig. 1 is shown in Fig. 2.

An example of each class in Lemma 1 when $n = 8$ is shown in Fig. 3.

3. A Lower Bound

In this section, we focus mainly on the positive part. The same discussion is possible for the negative part.

Given an assignment $\mathcal{A}$, consider a pair of two adjacent intervals. If one interval is contained in the other, the quantity of the cross-talk between them is the length of the shorter one. Hence, if $O^+(I_i) \leqq O^+(I_j)$ then the quantity of the cross-talk between them in the positive part is $|I_i^+ \wedge I_j^+| = |I_i^+|$. In this case, I_i is said to *dominate* I_j in the positive part in $\mathcal{A}$. An interval can dominate at most two intervals in $\mathcal{A}$ and the number of adjacent pairs is $n - 1$.

From these observations, we have that

Theorem 1: If n is odd,

$$\mathrm{Xtalk}^+(\mathcal{A}) \geqq 2 \left(\sum_{I \in \mathcal{S}_{s/+}} |I^+| \right),$$

If n is even,

$$\mathrm{Xtalk}^+(\mathcal{A}) \geqq 2 \left(\sum_{I \in \mathcal{S}_{s/+}} |I^+| \right) + |I^+_{\alpha/+}|.$$

□

Let p_i be the number of intervals dominated by $I_{i/+}$ in the positive part in $\mathcal{A}$. Trivially, p_i is either 0, 1, or 2, and their sum is $n - 1$. The sequence $p(\mathcal{A}) =$

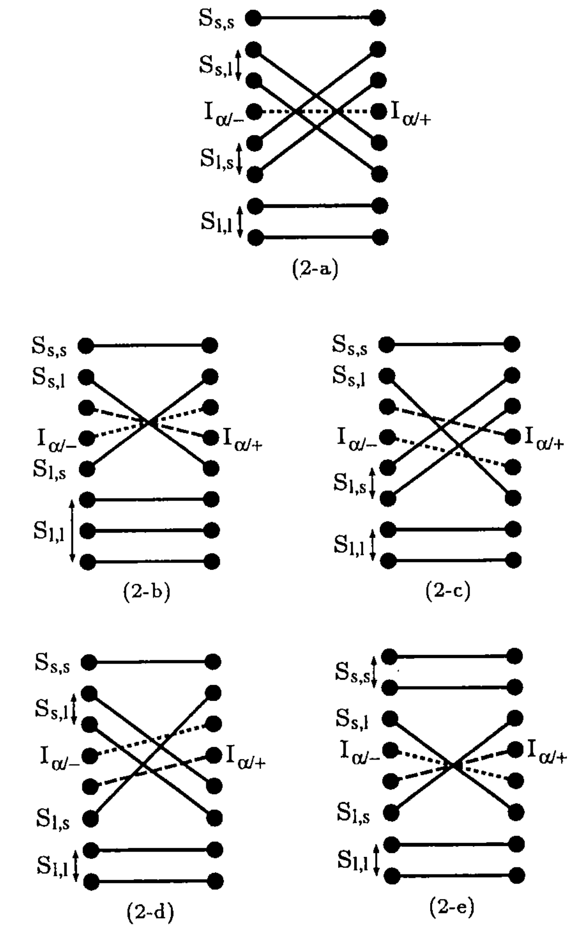

Fig. 3 Typical interval sets belonging to each class. (dotted lines: $I_{\alpha/+}$, broken lines: $I_{\alpha/-}$)

$(p_1, p_2, \ldots, p_n)$ is called the positive dominating vector of $\mathcal{A}$. The total cross-talk of $\mathcal{A}$ in the positive part is $\sum_{i=1}^{n} p_i |I^+_{i/+}|$.

The set of positive dominating vectors is a partial ordered set with respect to the total cross-talk in the positive part in the following sense.

Lemma 2: $\mathrm{Xtalk}^+(\mathcal{A}_1) \leqq \mathrm{Xtalk}^+(\mathcal{A}_2)$ if

$$p(\mathcal{A}_1) = (p_1, \ldots, p_i, \ldots, p_j, \ldots, p_n)$$

and

$$p(\mathcal{A}_2) = (p_1, \ldots, p_i - 1, \ldots, p_j + 1, \ldots, p_n)$$

$(1 \leqq i < j \leqq n)$. □

Proof: Since $|I^+_{j/+}| \geqq |I^+_{i/+}|$, $\mathrm{Xtalk}^+(\mathcal{A}_2) - \mathrm{Xtalk}^+(\mathcal{A}_1) = |I^+_{j/+}| - |I^+_{i/+}| \geqq 0$. □

An assignment $\mathcal{A}$ and the positive dominating vector of $\mathcal{A}$ are called *minimal* in the positive part if the positive dominating vector of $\mathcal{A}$ is

$$(\underbrace{2, 2, \ldots, 2}_{\alpha}, 0, 0, \ldots, 0)$$

when n is odd,

$$(\underbrace{2,2,\ldots,2}_{\alpha-1},1,0,0,\ldots,0)$$

when n is even.

Since $\text{Xtalk}^+(\mathcal{A}) = \sum_{i=1}^{n} p(i)|I_i^+|$, the following lemma is trivial.

Lemma 3: If an assignment $\mathcal{A}$ is minimal in the positive part, $\mathcal{A}$ attains the lower bound in Theorem 1 in the positive part. □

If there is no interval which has the same length as $I_{\alpha/+}$ in the positive part, the condition in Lemma 3 is not only sufficient but also necessary.

An assignment $\mathcal{A}$ is called *alternate* in the positive part if

- An interval of class $\mathcal{S}_{l/+}$ is not adjacent to any of class $\mathcal{S}_{l/+}$, and
- if n is even, $I_{\alpha/+}$ is not adjacent to two intervals of class $\mathcal{S}_{l/+}$.

If n is odd, the alternate assignment is characterized more definitely as $\mathcal{A}(I)$ is even for $I \in \mathcal{S}_{s/+}$ and $\mathcal{A}(I)$ is odd for $I \in \mathcal{S}_{l/+}$.

Lemma 4: An assignment $\mathcal{A}$ is alternate in the positive part if and only if the positive dominating vector of $\mathcal{A}$ is minimal. □

Proof: To prove Lemma 4, first, we assume that $\mathcal{A}$ is alternate in the positive part. Since $|\mathcal{S}_{l/+}| = \lceil \frac{n}{2} \rceil$ and each interval of $\mathcal{S}_{l/+}$ is not adjacent to the others, if n is odd, each interval of $\mathcal{S} - \mathcal{S}_{l/+} = \mathcal{S}_{s/+}$ is not adjacent to the others, or if n is even, at most two intervals of $\mathcal{S} - \mathcal{S}_{l/+} = \mathcal{S}_{s/+} \cup \{I_{\alpha/+}\}$ are adjacent to each other. If n is odd, each interval of $\mathcal{S}_{s/+}$ dominates two intervals in $\mathcal{A}$ in the positive part. Thus, the positive dominating vector of $\mathcal{A}$ is minimal. If n is even, there can exist two following cases. First, we consider the case that intervals of $\mathcal{S} - \mathcal{S}_{l/+} = \mathcal{S}_{s/+} \cup \{I_{\alpha/+}\}$ are not adjacent to each other. In this case, $I_{\alpha/+}$ is assigned either track 1, or n, since $I_{\alpha/+}$ is not adjacent to two intervals of class $\mathcal{S}_{l/+}$, and dominates one interval in $\mathcal{A}$ in the positive part. And each interval of $\mathcal{S}_{s/+}$ dominates two intervals in $\mathcal{A}$ in the positive part. Thus, the positive dominating vector of $\mathcal{A}$ is minimal. Second, we consider the case that two intervals of $\mathcal{S} - \mathcal{S}_{l/+} = \mathcal{S}_{s/+} \cup \{I_{\alpha/+}\}$ are adjacent to each other. In this case, intervals of $\mathcal{S}_{l/+}$ are assigned to track 1 and n, since $|\mathcal{S}_{l/+}| = |\mathcal{S}_{s/+} \cup \{I_{\alpha/+}\}|$. If intervals of $\mathcal{S}_{s/+}$ are adjacent, then $I_{\alpha/+}$ is adjacent to two intervals of class $\mathcal{S}_{l/+}$. Thus, an interval of $\mathcal{S}_{s/+}$ and $I_{\alpha/+}$ are adjacent. Hence, $I_{\alpha/+}$ dominates one interval and each interval of $\mathcal{S}_{s/+}$ dominates two intervals in $\mathcal{A}$ in the positive part. Thus, the positive dominating vector of $\mathcal{A}$ is minimal.

Next, we assume that $\mathcal{A}$ is not alternate in the positive part. From the definition of alternate assignment, an interval of $\mathcal{S}_{l/+}$ dominates at least one interval in $\mathcal{A}$ or if n is even, $I_{\alpha/+}$ dominates two intervals in $\mathcal{A}$. Thus, the positive dominating vector of $\mathcal{A}$ is not minimal. □

An assignment which is alternate simultaneously in the positive part and negative part is called a *perfect assignment.* It is true that a perfect assignment, if one exists, attains the minimum total cross-talks over all assignments of $\mathcal{S}$. If there is no interval which has the same length as $I_{\alpha/+}$ in the positive part or $I_{\alpha/-}$ in the negative part, the assignment which attains the minimum total cross-talks over all assignments of $\mathcal{S}$ is only perfect. Though the number of alternate assignments in each part is so many as

$$\left(\frac{n-1}{2}\right)! \cdot \left(\frac{n+1}{2}\right)! \quad \text{if} \quad n \text{ is odd,}$$
$$n \cdot \left(\frac{n-2}{2}\right)! \cdot \left(\frac{n}{2}\right)! \quad \text{if} \quad n \text{ is even,}$$

it is not a trivial problem for given $\mathcal{S}$ to answer if there exists a perfect assignment.

4. Intersection Sets with Perfect Assignment

A characterization of the interval set which has a perfect assignment is given in a simple form but its verification needs a lengthy case study.

Theorem 2: The necessary and sufficient condition of an interval set $\mathcal{S}$ to have a perfect assignment is that there exists a classification of $\mathcal{S}$ into $\mathcal{S}_{s,s}$, $\mathcal{S}_{s,l}$, $\mathcal{S}_{l,s}$, and $\mathcal{S}_{l,l}$ such that

(1) if n is odd, $\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} = \emptyset$;

(2) if n is even, $(I_{\alpha/+} \neq I_{\alpha/-})$ or $(\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} = \emptyset)$. □

Proof when n is odd

First, we prove the sufficiency. Since $\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} = \emptyset$, $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$ cover $\mathcal{S}$. Furthermore, it holds by Lemma 1 that $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}| + 1$. Therefore, the assignment $\mathcal{A}$ as follows is possible.

> Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$ alternately to the tracks from the first track.

Since $\mathcal{S}_{l,l} \in \mathcal{S}_{l/-} \cap \mathcal{S}_{l/+}$ and $\mathcal{S}_{s,s} \in \mathcal{S}_{s/-} \cap \mathcal{S}_{s/+}$, an interval of class $\mathcal{S}_{l/-}$ is not adjacent to any of class $\mathcal{S}_{l/-}$, nor is an interval of class $\mathcal{S}_{l/+}$ adjacent to any of class $\mathcal{S}_{l/+}$. Thus, $\mathcal{A}$ is perfect.

Next, the necessity is shown as follows: Assume without loss of generality that $\mathcal{S}_{l,s} = \mathcal{S}_{l/-} \cap \mathcal{S}_{s/+} \neq \emptyset$. If $\mathcal{A}$ is a perfect assignment, $\mathcal{A}(I)$ is odd for $I \in \mathcal{S}_{l/-}$ and $\mathcal{A}(I)$ is even for $I \in \mathcal{S}_{s/+}$. This is impossible since there is an interval that belongs to $\mathcal{S}_{l,s}$.

Proof when n is even

To prove the sufficiency, we define the assignment $\mathcal{A}$ for the cases (2-a)~(2-e) which were defined in Lemma 1. The proof of the perfectness of each case is clear from the definition.

- case (2-a)

Necessarily, $\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} = \emptyset$. So $\{I_{\alpha/+}\} \cup \mathcal{S}_{s,s} \cup \mathcal{S}_{l,l}$

covers all the intervals. Furthermore, since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}|+1$ by Lemma 1, the following perfect assignment is possible.

> Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately, from the 1st track to the $n-1$-th track, and let $\mathcal{A}(I_{\alpha/+}) = n$.

In cases (2-b)~(2-e), $I_{\alpha/+} \neq I_{\alpha/-}$.

- case (2-b)

  Since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}| + 2$ and $|\mathcal{S}_{s,l}| = |\mathcal{S}_{l,s}|$ by Lemma 1, the following assignment is possible.

  > Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately as possible, from the 1st track to $2|\mathcal{S}_{s,s}|+1$-th track. Then let $\mathcal{A}(I_{\alpha/+}) = 2|\mathcal{S}_{s,s}|+2$. Next, assign intervals of $\mathcal{S}_{l,s}$ and $\mathcal{S}_{s,l}$, alternately as possible, starting with an interval of $\mathcal{S}_{l,s}$. The last interval is assigned $n-2$. Let $\mathcal{A}(I_{\alpha/-}) = n-1$. Finally, for the remaining interval I, which is of $\mathcal{S}_{l,l}$, let $\mathcal{A}(I) = n$.

- case (2-c)

  Since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}| + 1$ and $|\mathcal{S}_{s,l}| = |\mathcal{S}_{l,s}| - 1$ by Lemma 1, the following assignment is possible.

  > Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately as possible, from the 1st track to, say, $2|\mathcal{S}_{s,s}| + 1$-th track. Then let $\mathcal{A}(I_{\alpha/+}) = 2|\mathcal{S}_{s,s}| + 2$. Next, assign intervals of $\mathcal{S}_{l,s}$ and $\mathcal{S}_{s,l}$, alternately as possible, starting with an interval of $\mathcal{S}_{l,s}$. The last interval is assigned $n-1$. Let $\mathcal{A}(I_{\alpha/-}) = n$.

- case (2-d)

  Since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}| + 1$ and $|\mathcal{S}_{s,l}| = |\mathcal{S}_{l,s}| + 1$ by Lemma 1, the following assignment is possible.

  > Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately as possible, from the 1st track to, say, $2|\mathcal{S}_{s,s}|$-th track. Then let $\mathcal{A}(I_{\alpha/+}) = 2|\mathcal{S}_{s,s}| + 1$. Next, assign intervals of $\mathcal{S}_{l,s}$ and $\mathcal{S}_{s,l}$, alternately as possible, starting with an interval of $\mathcal{S}_{l,s}$. The last interval is assigned $n-2$. Let $\mathcal{A}(I_{\alpha/-}) = n-1$. Finally, for the remaining interval I, which is of $\mathcal{S}_{l,l}$, let $\mathcal{A}(I) = n$.

- case (2-e)

  Since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}|$ and $|\mathcal{S}_{s,l}| = |\mathcal{S}_{l,s}|$ by Lemma 1, the following assignment is possible.

  > Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately as possible, from the 1st track to, say, $2|\mathcal{S}_{s,s}|$-th track. Then let $\mathcal{A}(I_{\alpha/+}) = 2|\mathcal{S}_{s,s}| + 1$. Next, assign intervals of $\mathcal{S}_{l,s}$ and $\mathcal{S}_{s,l}$, alternately as possible, starting with an interval of $\mathcal{S}_{l,s}$. The last interval is assigned $n-1$. Let $\mathcal{A}(I_{\alpha/-}) = n$.

To prove the necessity, assume that $I_{\alpha/+} = I_{\alpha/-}$ and $\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} \neq \emptyset$. This case falls in Case (2-a). Since $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}| + 1$ and an interval of class $\mathcal{S}_{l/-}$ is not adjacent to any of class $\mathcal{S}_{l/-}$, nor is an interval of class $\mathcal{S}_{l/+}$ adjacent to any of class $\mathcal{S}_{l/+}$ from the definition of alternate, the elements of $\mathcal{S}_{s,s} \cup \mathcal{S}_{l,l}$ are assigned alternately starting and ending with the intervals of $\mathcal{S}_{l,l}$ in both part in any perfect assignment. Since $\mathcal{S}_{l,s} \cup \mathcal{S}_{s,l} \neq \emptyset$, assume without loss of generality $I_i \in \mathcal{S}_{l,s}$. I_i is adjacent to some $I_j \in \mathcal{S}_{l,l}$. Then, I_i or I_j dominates the other in the negative part which violates the definition of alternate assignment. □

5. Intersection Sets with No Perfect Assignment

If an interval set has no perfect assignment, it satisfies the following conditions by Theorem 2.

1. If n is odd, $\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} \neq \emptyset$.
2. If n is even, $(I_{\alpha/+} = I_{\alpha/-})$ and $(\mathcal{S}_{s,l} \cup \mathcal{S}_{l,s} \neq \emptyset)$.

For description, certain particular dominating vectors in $\mathcal{A}$ are defined.

- If n is odd, $p(\mathcal{A}) = (p_i)$ is o-minimal if:

$$p(\mathcal{A}) = (\underbrace{2, \cdots, 2}_{\alpha-1}, 1, 1, 0, \cdots, 0).$$

- If n is even, $p = (p_i)$ is
  1. μ-minimal:
  $$p(\mathcal{A}) = (\underbrace{2, \cdots, 2}_{\alpha-2}, 1, 2, 0, \cdots, 0).$$
  2. ν-minimal:
  $$p(\mathcal{A}) = (\underbrace{2, \cdots, 2}_{\alpha-1}, 0, 1, 0, \cdots, 0).$$

The corresponding dominating vector in the negative part $n(\mathcal{A}) = (n_i)$ is analogously defined.

Theorem 3: If $\mathcal{S}$ does not have a perfect assignment, then it is possible to construct an assignment $\mathcal{A}$ of $\mathcal{S}$ which satisfies the following conditions. The assignment attains the minimum total cross-talk.

- When n is odd, one of $p(\mathcal{A})$ and $n(\mathcal{A})$ is minimal and another is o-minimal.

- When n is even, one of $p(\mathcal{A})$ and $n(\mathcal{A})$ is minimal and another is μ-minimal or ν-minimal.

□

Proof: Note that $\mathcal{A}$ is taken from two choices if n is odd, four choices if n is even. We have to prove that there are choices as in the theorem and that one of such assignment attains the minimum total cross-talk. From Lemma 2, the assignment $\mathcal{A}$ whose $p(\mathcal{A})$ is o-minimal attains the minimum total cross-talk in the positive part over all assignments of $\mathcal{S}$ except the assignment $\mathcal{A}'$ whose $p(\mathcal{A}')$ is minimal. And it is impossible to be minimal in both parts from the assumption. Thus, the latter is satisfied. The former will be proved by construction.

[**When n is odd**]

1. Construct $\mathcal{A}$ such that $p(\mathcal{A})$ is minimal and $n(\mathcal{A})$ is o-minimal.

   Starting with an interval of $\mathcal{S}_{l,l}$, assign intervals of $\mathcal{S}_{s,s}$ and $\mathcal{S}_{l,l}$, alternately as possible, except $\{I_{\alpha/-}, I_{(\alpha+1)/-}\}$. Note that $I_{\alpha/-}$ is included in $\mathcal{S}_{s,s}$ or $\mathcal{S}_{s,l}$ and that $I_{(\alpha+1)/-}$ is included in $\mathcal{S}_{l,s}$ or $\mathcal{S}_{l,l}$. Let the last interval be denoted by I^*. Next take one of $I_{\alpha/-}$ and $I_{(\alpha+1)/-}$ following the rule that

   if $I^* \in \mathcal{S}_{l,l}$, take the one of $\mathcal{S}_{s/+}$, or if $I^* \in \mathcal{S}_{s,s}$, take the one of $\mathcal{S}_{l/+}$.

   (Note that $I_{\alpha/-}$ or $I_{(\alpha+1)/-}$ belongs to $\mathcal{S}_{s/+}$ in the case of $I^* \in \mathcal{S}_{l,l}$. Otherwise, it holds that $I^* \in \mathcal{S}_{l,l}$, $I_{\alpha/-} \in \mathcal{S}_{s,l}$, and $I_{(\alpha+1)/-} \in \mathcal{S}_{l,l}$. Then $|\mathcal{S}_{l,l}| = |\mathcal{S}_{s,s}|+2$, a contradiction. The analogous fact holds in the case of $I^* \in \mathcal{S}_{s,s}$.)

   Then, one interval remains which is $I_{\alpha/-}$ or $I_{(\alpha+1)/-}$, which we denote I'. Restart the assignment of $\mathcal{S}_{l,s}$ and $\mathcal{S}_{s,l}$, alternately as possible, except I', starting with an interval such that it belongs to the different subset in the negative part and the same subset in the positive part with respect to I^*. (For example, if $I^* \in \mathcal{S}_{l,l}$, then the next interval is from $\mathcal{S}_{s,l}$.)

   At this stage, I' and possibly one interval of $\mathcal{S}_{l,l}$ will remain. Take I' and then assign the remaining interval.

   It is clear that $p(\mathcal{A})$ is minimal since we are always following the strategy to change the subsets from "$l/+$" and "$s/+$" in the positive part. In the negative part, the procedure goes the same way with exception when $I_{\alpha/-}$ or $I_{(\alpha+1)/-}$ is assigned. This causes the number of dominating by them being exactly 1, to lead to

$$n(\mathcal{A}) = (\underbrace{2, \cdots, 2}_{\alpha-1}, 1, 1, 0, \cdots, 0).$$

2. Construct $\mathcal{A}$ such that $p(\mathcal{A})$ is o-minimal and $n(\mathcal{A})$ is minimal.

   The construction is the same as above exchanging '$-$' and '$+$.'

[**When n is even**]

The construction of $\mathcal{A}$ is by a tricky way.

Suppose we want to have an assignment $\mathcal{A}$ such that $n(\mathcal{A})$ is minimal and $p(\mathcal{A})$ is μ-minimal. Then, alter the intervals $I_{\alpha/+}$ and $I_{(\alpha-1)/+}$ by exchanging positive parts $I^+_{\alpha/+}$ and $I^+_{(\alpha-1)/+}$ to get new set $\mathcal{S}'$. Since $\mathcal{S}'$ violates the condition $I_{\alpha/+} = I_{\alpha/-}$, there exists a perfect assignment. Get a perfect assignment of $\mathcal{S}'$, and keeping its order recover the interval set $\mathcal{S}$. Then, the assignment $\mathcal{A}$ of $\mathcal{S}$ will be such that $n(\mathcal{A})$ is minimal and

$$p(\mathcal{A}) = (\underbrace{2, \cdots, 2}_{\alpha-2}, 1, 2, 0, \cdots, 0).$$

Suppose we want have an assignment $\mathcal{A}$ such that $n(\mathcal{A})$ is minimal and $p(\mathcal{A})$ is ν-minimal. Then we have only to change $I_{\alpha/+}$ and $I_{(\alpha+1)/+}$.

Other symmetric cases are omitted. □

Finally, we summarize the procedures into one (see Fig. 4).

Procedure: Min Total Xtalk($\mathcal{S}$)
Step 1. Determine if $\mathcal{S}$ has a perfect assignment or not by the condition presented in Theorem 2.
Step 2. If yes, apply the algorithms presented in the proof of Theorem 2.
Step 3. Otherwise,
3-1. Construct two assignments (if n is odd) or four assignments (if n is even) by the way presented in the proof of Theorem 3.
3-2. Compute the total cross-talks of them and take the one with the minimum.

Fig. 4 Min Total Xtalk ($\mathcal{S}$).

Theorem 4: The computational complexity of **Min Total Xtalk($\mathcal{S}$)** is $O(n \log n)$ where $n = |\mathcal{S}|$.

6. Conclusions

This paper solved a total cross-talk minimization problem. The model assumed is very basic, and the result looks too much theoretical. The future works will be to extend the result to meet the practical situations as follows:

1. (Tapered cross-talk) The cross-talk is defined on each interval as the weighted sum of cross-talks affected from other intervals not only the one from the adjacent ones.

2. (Maximum cross-talk) Evaluation is the maximum of cross-talks between adjacent intervals,
3. (Non-intersecting intervals) Intervals which are not intersecting are allowed,
4. (Cross-talk limitation) For each pair of intervals, the bound of admissible cross-talk is preassigned,
5. (Individual cross-talk) For each pair of intervals, the cross-talk when they are adjacent is preassigned,
6. (Rectilinear intervals) Cross-talks caused by vertical connections to the terminals are involved.

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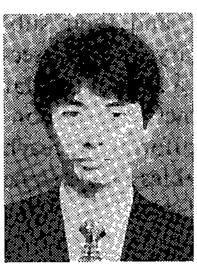

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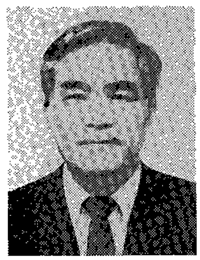

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