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Fault-Tolerant Graphs for Hypercubes and Tori*

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SUMMARY Motivated by the design of fault-tolerant multiprocessor interconnection networks, this paper considers the following problem: Given a positive integer t and a graph H, construct a graph G from H by adding a minimum number $\Delta(t, H)$ of edges such that even after deleting any t edges from G the remaining graph contains H as a subgraph. We estimate $\Delta(t, H)$ for the hypercube and torus, which are well-known as important interconnection networks for multiprocessor systems. If we denote the hypercube and the square torus on N vertices by Q_N and D_N respectively, we show, among others, that $\Delta(t, Q_N) = O(tN \log(\frac{\log N}{t} + \log 2e))$ for any t and N $(t \geqq 2)$, and $\Delta(1, D_N) = \frac{N}{2}$ for N even.

key words: *hypercubes, tori, edge-fault-tolerant graphs, matric graphs, error-correcting binary linear codes*

1. Introduction

Motivated by the design of fault-tolerant multiprocessor interconnection networks, this paper considers the following problem: Given a positive integer t and a graph H, construct a graph G from H by adding a minimum number of edges such that even after deleting any t edges from G the remaining graph contains H as a subgraph. We construct such graphs by adding small number of edges for the hypercube and torus, which are well-known as important interconnection networks for multiprocessor systems. Many related results can be found in the literature [1]–[15], [17]–[23], [25]–[29].

Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. If $S \subseteqq E(G)$, $G \setminus S$ is the graph obtained from G by deleting the edges of S.

Let t be a positive integer. A graph G is called a t-EFT (t-edge-fault-tolerant) graph for a graph H if $G \setminus S$ contains H as a subgraph for every $S \subseteqq E(G)$, with $|S| \leqq t$. Let $\Delta(t, H)$ denote the minimum number of edges added to H to construct a t-EFT graph for H with $|V(H)|$ vertices.

Let P_N, C_N, L_N, D_N, B_N, and Q_N denote the path, cycle, square grid, square torus, complete binary tree, and hypercube on N vertices, respectively. The following results can be found in the literature.

Theorem I: [23], [27] $\Delta(t, P_N) = \lceil \frac{1}{2}(t-1)N \rceil + 1$ $(t \leqq N-2)$.

Theorem II: [23], [27] $\Delta(t, C_N) = \lceil \frac{1}{2}tN \rceil$ $(t \leqq N-3)$.

Theorem III: [18] $\Delta(1, L_N) \leqq 2\sqrt{N}$.

Theorem IV: [26] $\frac{3}{8}(N+1) \leqq \Delta(1, B_N) \leqq \frac{11}{16}(N+1) - 2$.

Theorem V: [18], [21], [26] $\Delta(1, Q_N) = \frac{N}{2}$.

In this paper, we generalize Theorem V and show that

$$\Delta(t, Q_N) = O\left(tN \log\left(\frac{\log N}{t} + c\right)\right) \quad (1)$$

for any t and N $(t \geqq 2)$, where $c = \log 2e$. We also show that $\Delta(1, D_N) = \frac{N}{2}$ if N is even. Some generalizations of the latter result are also mentioned.

We introduce in Sect. 2 matric graphs associated with binary matrices as a generalization of hypercubes. The upper bound of $\Delta(t, Q_N)$ in (1) is proved by constructing a t-EFT matric graph for Q_N associated with a basis matrix of an error-correcting binary linear code. It is interesting that the t-EFT matric graph for Q_N proposed here has a strong fault-tolerance property. We show that even after deleting $\frac{1}{2}tN$ edges of t different dimensions from a t-EFT matric graph for Q_N, the remaining graph still contains Q_N as a subgraph. An essentially same construction is proposed in [13] independent of this paper.

2. Hypercubes and Matric Graphs

The n-cube (n-dimensional cube), denoted by $Q(n)$, is defined as follows: $V(Q(n)) = \{0,1\}^n$; $E(Q(n)) = \{(u,v) | u, v \in V(Q(n)), w(u \oplus v) = 1\}$, where $\oplus$ denotes bit-wise addition modulo 2 and $w(x)$ is the Hamming weight of vector x, that is the number of 1's which x contains. It is easy to see that $Q(n)$ is connected and $|V(Q(n))| = 2^n$. Since the degree of each vertex of $Q(n)$ is n, $|E(Q(n))| = n2^{n-1}$. An edge (x, y) is called an i-edge (i-dimensional edge) if x and y differ in only the i-th bit. The number of i-edges of $Q(n)$ is 2^{n-1}. It is easy to see that the graph obtained from $Q(n)$ by deleting all i-edges is consisting of two disjoint copies of $(n-1)$-cubes. A graph G is called a hypercube if G is isomorphic to $Q(n)$ for some n.

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Let M be an (m,n)-binary matrix, which is an m by n matrix consisting of 0's and 1's. Let r_i and c_j denote the i-th row and the j-th column of M, respectively. Define $R(M) = \{r_1, r_2, \ldots, r_m\}$ and $C(M) = \{c_1, c_2, \ldots, c_n\}$.

The matric graph associated with an (m,n)-binary matrix M, denoted by $G(M)$, is defined as follows: $V(G(M)) = \{0,1\}^n$; any two vertices u and v are joined by $|\{r \in R(M) | r = u \oplus v\}|$ parallel edges. An edge (u,v) of $G(M)$ is said to be of dimension $r(\in R(M))$ if $r = u \oplus v$. For $r \in R(M)$, $E(r)$ is the set of all edges of dimension r of $G(M)$. For $S \subseteqq R(M)$, $E(S) = \bigcup_{r \in S} E(r)$. Since each vertex of $G(M)$ is incident to an edge of dimension r for any $r \in R(M)$, the degree of each vertex of $G(M)$ is m. Thus $|E(G(M))| = m2^{n-1}$. For $S \subseteqq R(M)$, let $M \setminus S$ denote the matrix obtained from a binary matrix M by deleting the rows of S. It is easy to see the following two lemmas from the definition of the matric graph.

Lemma 1: If I_n is the (n,n)-unit matrix, $G(I_n) = Q(n)$. Moreover the edges of dimension r_i of $G(I_n)$ correspond to the i-edges of $Q(n)$.

Lemma 2: $G(M \setminus S) = G(M) \setminus E(S)$.

Lemma 3: If a binary matrix M has a column consisting of 0's, $G(M)$ is disconnected.

Proof: Assume that $c_i = 0$ for some i. Define $V_0 = \{v \in V(G(M)) | v_i = 0\}$ and $V_1 = \{v \in V(G(M)) | v_i = 1\}$, where v_i denotes the i-th bit of v. (V_0, V_1) is a partition of $V(G(M))$. Since $c_i = 0$, there exists no edge joining a vertex in V_0 and a vertex in V_1. Hence, $G(M)$ is disconnected. □

Lemma 4: If M' is a matrix obtained from an (m,n)-binary matrix M by elementary column operations, $G(M')$ is isomorphic to $G(M)$.

Proof: It suffices to prove the following: (mcxlvii) If M' is a matrix obtained from M by exchanging column c_{j_1} with column c_{j_2}, $1 \leqq j_1 < j_2 \leqq n$, then $G(M')$ is isomorphic to $G(M)$; (mcxlviii) If M' is a matrix obtained from M by adding c_{j_2} to c_{j_1}, $j_1 \neq j_2$, then $G(M')$ is isomorphic to $G(M)$.

Proof of (mcxlvii): Let φ_1 be a mapping of $V(G(M))$ to $V(G(M'))$ such that $\varphi_1(v) = (v_1, \ldots, v_{j_1-1}, v_{j_2}, v_{j_1+1}, \ldots, v_{j_2-1}, v_{j_1}, v_{j_2+1}, \ldots, v_n)$. If $\varphi_1(u) = \varphi_1(v)$, then $u_j = v_j (1 \leqq j \leqq n)$, and so $u = v$. Thus, φ_1 is a one-to-one mapping. Since $|V(G(M))| = |V(G(M'))| = 2^n$, φ_1 is a bijection. Let r'_i denote the i-th row of M'. If $r_i = (x_1, x_2, \ldots, x_n)$, $r'_i = (x_1, \ldots, x_{j_1-1}, x_{j_2}, x_{j_1+1}, \ldots, x_{j_2-1}, x_{j_1}, x_{j_2+1}, \ldots, x_n)$. Since $u \oplus v = r_i$ if and only if $\varphi_1(u) \oplus \varphi_1(v) = r'_i$, $(u,v) \in E(G(M))$ if and only if $(\varphi_1(u), \varphi_1(v)) \in E(G(M'))$. Thus $G(M')$ is isomorphic to $G(M)$.

Proof of (mcxlviii): Let φ_2 be a mapping of $V(G(M))$ to $V(G(M'))$ such that $\varphi_2(v) = (v_1, \ldots, v_{j_1-1}, v_{j_1} \oplus v_{j_2}, v_{j_1+1}, \ldots, j_n)$. If $\varphi_2(u) = \varphi_2(v)$, then $u_j = v_j$ $(j \neq j_1)$ and $u_{j_1} \oplus u_{j_2} = v_{j_1} \oplus v_{j_2}$. Since $u_{j_2} = v_{j_2}$, we obtain $u_{j_1} = v_{j_1}$, and so $u = v$. Thus φ_2 is a bijection. If $r_i = (x_1, x_2, \ldots, x_n)$, $r'_i = (x_1, \ldots, x_{j_1-1}, x_{j_1} \oplus x_{j_2}, x_{j_1+1}, \ldots, x_n)$. Since $u \oplus v = r_i$ if and only if $\varphi_2(u) \oplus \varphi_2(v) = r'_i$, $(u,v) \in E(G(M))$ if and only if $(\varphi_2(u), \varphi_2(v)) \in E(G(M'))$. Thus $G(M')$ is isomorphic to $G(M)$. □

Theorem 1: For any (n,n)-binary matrix M, $G(M)$ is isomorphic to $Q(n)$ if and only if M is non-singular.

Proof: If M is non-singular, we can obtain the unit matrix I_n from M by elementary column operations. Thus, $G(M)$ is isomorphic to $Q(n)$ by Lemmas 1 and 4.

If M is singular, then we can obtain a matrix with a column consisting of 0's from M by elementary column operations. Thus, $G(M)$ is not isomorphic to $Q(n)$, since $G(M)$ is disconnected by Lemmas 3 and 4. □

Corollary 1: For any (m,n)-binary matrix M, $G(M)$ contains $Q(n)$ as a subgraph if and only if the rank of M is n.

Proof: If the rank of M is n, then there exists $S \subset R(M)$ with $|S| = m - n$ such that $M \setminus S$ is an (n,n)-binary non-singular matrix. Thus, $G(M \setminus S)$ is isomorphic to $Q(n)$ by Theorem 1, and so $G(M)$ contains $Q(n)$ as a subgraph by Lemma 2.

If the rank of M is less than n, we can obtain a matrix with a column consisting of 0's from M by elementary column operations. Thus, $G(M)$ is disconnected by Lemmas 3 and 4. Since $|V(G(M))| = |V(Q(n))| = 2^n$, we conclude that $G(M)$ does not contain $Q(n)$ as a subgraph. □

3. t-DFT Matric Graphs for Hypercubes

Let M be an (m,n)-binary matrix. $G(M)$ is called a t-DFT (t-dimension-fault-tolerant) matric graph for $Q(n)$ if $G(M) \setminus E(S) (= G(M \setminus S))$ contains $Q(n)$ as a subgraph for any $S \subseteqq R(M)$, with $|S| \leqq t$. Define $\Lambda(t,n) = \min\{|E(G(M))| - |E(Q(n))| \mid G(M) :$ t-DFT matric graph for $Q(n)\}$. Since the degree of each vertex of $G(M)$ is m, the problem of finding $\Lambda(t,n)$ is equivalent to the one of finding the minimum number of rows of a binary matrix M such that $G(M)$ is a t-DFT matric graph for $Q(n)$. The following theorem characterizes the t-DFT matric graph for $Q(n)$.

Theorem 2: For any (m,n)-binary matrix M, $G(M)$ is a t-DFT matric graph for $Q(n)$ if and only if the Hamming weight of any linear combination of $C(M)$ is at least $t+1$.

Proof: Assume that there exists a linear combination of $C(M)$ such that its Hamming weight is at most t. Then we can obtain a matrix M' with a column (say, the j-th column) of Hamming weight at most t from M by some elementary column operations. Let S be the set of rows of M corresponding to the rows S' of M' whose j-th bits are 1's. Since the j-th column of $M' \setminus S'$ is consisting of 0's, $G(M' \setminus S')$ does not contain $Q(n)$

as a subgraph by Corollary 1. Since $G(M \setminus S)$ is isomorphic to $G(M' \setminus S')$ by Lemma 4, and $|S| = |S'| \leqq t$, we conclude that $G(M)$ is not a t-DFT matric graph for $Q(n)$.

Conversely, assume that the Hamming weight of any linear combination of $C(M)$ is at least $t+1$. Then $C(M \setminus S)$ is linearly independent for any $S \subseteqq R(M)$ with $|S| \leqq t$, and the rank of $M \setminus S$ is n. Thus $G(M \setminus S)$ contains $Q(n)$ as a subgraph by Corollary 1, and we conclude that $G(M)$ is a t-DFT matric graph for $Q(n)$. □

Theorem 2 means that for any (m,n)-binary matrix M, $G(M)$ is a t-DFT matric graph for $Q(n)$ if and only if $C(M)$ is a basis of an n-dimensional binary vector space such that the Hamming weight of any non-zero vector is at least $t+1$. Thus t-DFT matric graphs can be characterized by error-correcting binary linear codes.

Theorem 3: For any (m,n)-binary matrix M, $G(M)$ is a t-DFT matric graph for $Q(n)$ if and only if $C(M)$ is a basis of an n-dimensional binary linear code with minimum distance at least $t+1$.

The following bounds for the existence of n-dimensional binary linear codes with minimum distance at least $t+1$ are well-known. (See [24], for example.)

Theorem VI: If there exists an n-dimensional binary linear code with minimum distance at least $t+1$ and length m, then

$$2^{m-n} \geqq \sum_{i=0}^{\lfloor \frac{t}{2} \rfloor} \binom{m}{i}.$$

Theorem VII: There exists an n-dimensional binary linear code with minimum distance at least $t+1$ and length m for which the following inequality holds:

$$2^{m-n} \leqq \sum_{i=0}^{t-1} \binom{m}{i}.$$

The inequalities of Theorems VI and VII are well-known as Hamming bound and Varsharmov-Gilbert bound, respectively. It should be noted that Theorem VII is proved constructively. In what follows, we estimate $\Lambda(t,n)$ from Theorems 3, VI, and VII. We need a few lemmas.

Lemma 5: For $1 \leqq k \leqq m$,

$$\left(\frac{m}{k}\right)^k \leqq \sum_{i=0}^{k} \binom{m}{i} \leqq 2^k \binom{m}{k}$$

Proof: Let $S(m,k) = \sum_{i=0}^{k} \binom{m}{i}$. Then

$$S(m,k) \geqq \binom{m}{k} = \frac{m(m-1)\cdots(m-k+1)}{k!} \geqq \left(\frac{m}{k}\right)^k.$$

The right inequality is proved by induction on m and k.

(1) Since $S(m,1) = m+1 \leqq 2m$ and $S(k,k) = 2^k$, the claim is true if $k=1$ or $m=k$.

(2) Let $2 \leqq k < m$, and assume that the claim is true for $S(m,k')$, $S(m'k)$, and $S(m',k')$ with $m > m'$ and $k > k'$. Since

$$\binom{m}{i} = \binom{m-1}{i} + \binom{m-1}{i-1}$$

$$(1 \leqq i \leqq m-1),$$

$S(m,k) = S(m-1,k) + S(m-1,k-1)$. Thus,

$$S(m,k) \leqq 2^k \binom{m-1}{k} + 2^{k-1}\binom{m-1}{k-1} \leqq 2^k \left\{ \binom{m-1}{k} + \binom{m-1}{k-1} \right\} = 2^k \binom{m}{k}.$$

□

Lemma 6: [16] For any positive integer m and k where $m \geqq k$,

$$\binom{m}{k} \leqq \left(\frac{em}{k}\right)^k.$$

Lemma 7: Let $y \geqq 2$. If $x - \log x \leqq y$, then $x \leqq y + 2\log y$.

Proof: Assume that $x > y + 2\log y \geqq 2$ and let $g(z) = z - \log z$. Since $g(z)$ is an increasing function for $z > 2$, and $x > y + 2\log y > 2$,

$$g(x) - y > g(y + 2\log y) - y = 2\log y - \log(y + 2\log y) = \log \frac{y^2}{y + 2\log y}$$

Since $y^2 \geqq y + 2\log y$ for any $y \geqq 2$, $x - \log x > y$, which is a contradiction. □

Theorem 4: Let M be an (m,n)-binary matrix. If $G(M)$ is a t-DFT matric graph for $Q(n)$ $(t \geqq 2)$, then

$$m \geqq n + \left\lfloor \frac{t}{2} \right\rfloor \log\left(n / \left\lfloor \frac{t}{2} \right\rfloor\right).$$

Proof: If $G(M)$ is a t-DFT matric graph for $Q(n)$, $C(M)$ is a basis of an n-dimensional binary linear code with minimum distance at least $t+1$ by Theorem 3. Thus, by Theorem VI and Lemma 5, we have

$$2^{m-n} \geqq \sum_{i=0}^{k} \binom{m}{i} \geqq \left(\frac{m}{k}\right)^k,$$

where $k = \lfloor \frac{t}{2} \rfloor$. Hence

$$m - n \geqq k \log \frac{m}{k} \geqq k \log \frac{n}{k}.$$

□

Theorem 5: There exists an (m,n)-binary matrix M such that $G(M)$ is a t-DFT matric graph for $Q(n)$ $(t \geqq 2)$ and

$$m \leqq n + c(t-1) + 2(t-1)\log\left(\frac{n}{t-1} + c\right),$$

where $c = \log 2e$.

Proof: By Theorems 3 and VII, there exists an (m,n)-binary matrix M such that $G(M)$ is a t-DFT matric graph for $Q(n)$ $(t \geq 2)$ and $2^{m-n} \leq \sum_{i=0}^{t-1} \binom{m}{i}$. Thus, by Lemmas 5 and 6,

$$2^{m-n} \leqq \left(\frac{2em}{t-1}\right)^{t-1},$$

i.e.

$$m - n \leqq (t-1)\left(\log \frac{m}{t-1} + \log 2e\right),$$

i.e.

$$\frac{m}{t-1} - \log \frac{m}{t-1} \leqq \frac{n}{t-1} + c.$$

Since $c > 2$, by putting $x = \frac{m}{t-1}$ and $y = \frac{n}{t-1} + c$ in Lemma 7, we obtain

$$\frac{m}{t-1} \leqq \frac{n}{t-1} + c + 2\log\left(\frac{n}{t-1} + c\right).$$

Hence

$$m \leqq n + c(t-1) + 2(t-1)\log\left(\frac{n}{t-1} + c\right).$$

□

Since the numbers of edges of $G(M)$ and $Q(n)$ are $m2^{n-1}$ and $n2^{n-1}$, respectively, we obtain the following theorem from Theorems 4 and 5.

Theorem 6:

$$\left\lfloor \frac{t}{2} \right\rfloor 2^{n-1} log\left(n/\left\lfloor \frac{t}{2} \right\rfloor\right) \leqq \Lambda(t,n) \leqq (t-1)2^{n-1}\left\{2\log\left(\frac{n}{t-1} + c\right) + c\right\},$$

where $t \geq 2$ and $c = \log 2e$.

If $t < 2n$, bounds above are optimal to within a constant factor, and we have $\Lambda(t,n) = \Theta(t2^{n-1} \log(\frac{n}{t} + c))$. If $t \geqq 2n$, the upper bound above is $O(t2^{n-1}\log(\frac{n}{t} + c)) = O(t2^{n-1})$. We have a trivial lower bound of $t2^{n-1}$ since the degree of each vertex of a t-DFT matric graph is at least $n+t$. Thus our upper bound is also optimal to within a constant factor even if $t \geq 2n$. In summary, we have the following theorem.

Theorem 7: $\Lambda(t,n) = \Theta(t2^{n-1}\log(\frac{n}{t} + c))$ $(t \geqq 2, c = \log 2e)$.

4. t-EFT Graphs for Hypercubes

Since a t-DFT matric graph for $Q(n)$ is also a t-EFT graph for $Q(n)$, we have $\Delta(t, Q(n)) \leqq \Lambda(t,n)$. Thus we obtain the following theorem from Theorem 6

Theorem 8: $\Delta(t, Q(n)) = O(t2^{n-1}\log(\frac{n}{t} + c))$ $(t \geqq 2, c = \log 2e)$.

On the other hand, we have the following lower bound

$$\Delta(t, Q(n)) \geq t2^{n-1}, \tag{2}$$

since the degree of each vertex of a t-EFT graph for $Q(n)$ is at least $n+t$. It is an interesting open problem to close the gap between the bounds in Theorem 8 and (2). It should be noted that if $t = \Omega(n)$, then the upper bound in Theorem 8 is optimal to within a constant factor.

If M is the $(n+1,n)$-binary matrix obtained from I_n by adding a row consisting of 1's, then M satisfies the condition of Theorem 2 for $t=1$. Thus, we have

$$2^{n-1} \leqq \Delta(1, Q(n)) \leqq \Lambda(1,n) \leqq 2^{n-1},$$

which means Theorem V.

5. t-EFT Graphs for Tori

The Cartesian product of graphs G and H is the graph $G \times H$ with $V(G \times H) = V(G) \times V(H)$, in which (u,v) is adjacent to (u',v') if and only if either $u = u'$ and $(v,v') \in E(H)$ or $v = v'$ and $(u,u') \in E(G)$. $D(m_1, m_2, \ldots, m_d) = C_{m_1} \times C_{m_2} \times \cdots \times C_{m_d}$ is called a d-dimensional torus. We denote an element of $V(C_{m_1}) \times V(C_{m_2}) \times \cdots \times V(C_{m_d})$ by $\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_d \end{bmatrix}$, where $i_k \in V(C_{m_k})$ for $k = 1, 2, \ldots, d$.

Theorem 9: If m_1 and m_2 are even, $\Delta(1, D(m_1, m_2)) = \frac{N}{2}$, where $N = |V(D(m_1,m_2))| = m_1 \times m_2$.

Proof: Let $V(C_{m_k}) = \{0, 1, \ldots, m_k - 1\}$ for $k = 1, 2$. In what follows, we denote $D(m_1, m_2)$ by D.

Let G be the graph obtained from D by connecting $\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$ with $\begin{bmatrix} (i_1 + \frac{1}{2}m_1) \bmod m_1 \\ (i_2 + \frac{1}{2}m_2) \bmod m_2 \end{bmatrix}$ by an edge for $i_1 = 0, 1, \ldots, m_1 - 1, i_2 = 0, 1, \ldots, m_2 - 1$. It is easy to verify that $|E(G)| - |E(D)| = \frac{N}{2}$, since G is obtained from D by connecting each vertex with the unique farthest vertex in D by an edge. We will show that G is a 1-EFT graph for D.

Let e be any edge of G. If $e \notin E(D)$, it is trivial that $G \setminus \{e\}$ contains D as a subgraph. We show that $G \setminus \{e\}$ contains D as a subgraph for any $e \in E(D)$. We assume without loss of generality that $e = \left(\begin{bmatrix} 0 \\ m_2 - 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$. Let ϕ be a mapping from $V(D)$ to $V(G)$ such that

$$\phi : v \mapsto \begin{cases} v & \text{if } v \in V_1, \\ \begin{bmatrix} (i_1 + \frac{1}{2}m_1) \bmod m_1 \\ \frac{3}{2}m_2 - i_2 - 1 \end{bmatrix} & \text{if } v \in V_2, \end{cases}$$

where

$$v = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, \quad V_1 = \left\{ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} \;\middle|\; \begin{array}{l} 0 \leqq i_1 \leqq m_1 - 1, \\ 0 \leqq i_2 \leqq \frac{1}{2}m_2 - 1 \end{array} \right\},$$

and $V_2 = \{v \in V(D) \mid v \notin V_1\}$. Notice that (V_1, V_2) is a partition of $V(G)$. It is easy to see that $\phi(v) \in V_1$ if $v \in V_1$, and $\phi(v) \in V_2$ if $v \in V_2$. Thus $\phi(u) \neq \phi(v)$if$u \neq v$, which means that ϕ is a one-to-one mapping.

Now we show that $(\phi(u), \phi(v)) \in E(G) - \{e\}$ for any $(u, v) \in E(D)$. There are five cases.

Case 1) $u, v \in V_1$: In this case, $\phi(u) = u, \phi(v) = v$. Thus $(\phi(u), \phi(v)) \in E(G) - \{e\}$.

Case 2) $u = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, v = \begin{bmatrix} i_1 \\ i_2 + 1 \end{bmatrix}$ $(\frac{1}{2}m_2 \leqq i_2 \leqq m_2 - 2)$: In this case, $\phi(u) = \begin{bmatrix} i'_1 \\ i'_2 + 1 \end{bmatrix}, \phi(v) = \begin{bmatrix} i'_1 \\ i'_2 \end{bmatrix}$, where $i'_1 = (i_1 + \frac{1}{2}m_1) \bmod m_1, i'_2 = \frac{3}{2}m_2 - i_2 - 2$. Since $\frac{1}{2}m_2 \leqq i'_2 \leqq m_2 - 2$, $(\phi(u), \phi(v)) \in E(G) - \{e\}$.

Case 3) $u = \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}, v = \begin{bmatrix} (i_1 + 1) \bmod m_1 \\ i_2 \end{bmatrix}$ $(\frac{1}{2}m_2 \leqq i_2 \leqq m_2 - 1)$: In this case, $\phi(u) = \begin{bmatrix} i'_1 \\ i'_2 \end{bmatrix}, \phi(v) = \begin{bmatrix} (i'_1 + 1) \bmod m_1 \\ i'_2 \end{bmatrix}$, where $i'_1 = (i_1 + \frac{1}{2}m_1) \bmod m_1, i'_2 = \frac{3}{2}m_2 - i_2 - 1$. Thus $(\phi(u), \phi(v)) \in E(G) - \{e\}$.

Case 4) $u = \begin{bmatrix} i_1 \\ m_2 - 1 \end{bmatrix}, v = \begin{bmatrix} i_1 \\ 0 \end{bmatrix}$: In this case, $\phi(u) = \begin{bmatrix} (i_1 + \frac{1}{2}m_1) \bmod m_1 \\ \frac{1}{2}m_2 \end{bmatrix}, \phi(v) = \begin{bmatrix} i_1 \\ 0 \end{bmatrix}$. Thus $(\phi(u), \phi(v)) \in E(G) - \{e\}$.

Case 5) $u = \begin{bmatrix} i_1 \\ \frac{1}{2}m_2 - 1 \end{bmatrix}, \quad v = \begin{bmatrix} i_1 \\ \frac{1}{2}m_2 \end{bmatrix}$: In this case, $\phi(u) = \begin{bmatrix} i_1 \\ \frac{1}{2}m_2 - 1 \end{bmatrix}, \quad \phi(v) = \begin{bmatrix} (i_1 + \frac{1}{2}m_1) \bmod m_1 \\ m_2 - 1 \end{bmatrix}$. Thus $(\phi(u), \phi(v)) \in E(G) - \{e\}$.

Thus $(\phi(u), \phi(v)) \in E(G) - \{e\}$ for any $(u, v) \in E(D)$, and we conclude that $G \setminus \{e\}$ contains D as a subgraph. □

Theorem 9 can easily be generalized to the higher dimensional case.

Theorem 10: If m_k is even for any $k\,(k = 1, 2, \ldots, d)$, $\Delta(1, D(m_1, m_2, \ldots, m_d)) = \frac{N}{2}$, where $N = |V(D(m_1, m_2, \ldots, m_d))| = m_1 \times m_2 \times \cdots \times m_d$.

Proof: Let $V(C_{m_k}) = \{0, 1, \ldots, m_k - 1\}$ for $k = 1, 2, \ldots, d$. In what follows, we denote $D(m_1, m_2, \ldots, m_d)$ by D.

Let G be the graph obtained from D by connecting $\begin{bmatrix} i_1 \\ i_2 \\ \vdots \\ i_d \end{bmatrix}$ with $\begin{bmatrix} (i_1 + \frac{1}{2}m_1) \bmod m_1 \\ (i_2 + \frac{1}{2}m_2) \bmod m_2 \\ \vdots \\ (i_d + \frac{1}{2}m_d) \bmod m_d \end{bmatrix}$ by an edge for $i_k = 0, 1, \ldots, m_k - 1, k = 1, 2, \ldots, d$. It is easy to verify that $|E(G)| - |E(D)| = \frac{N}{2}$, since G is obtained from D by connecting each vertex with the unique farthest vertex in D by an edge. We can easily prove that G is a 1-EFT graph for D by a similar argument as the proof of Theorem 9, and we omit the details. □

We can prove that $\Delta(1, D(m_1, m_2, \cdots, m_d)) \leqq dN$ in general, and $\Delta(2, D(m_1, m_2)) \leqq \frac{3}{2}N$ for even $m_1, m_2 \geqq 6$, but the proofs are rather complicated and are omitted here.

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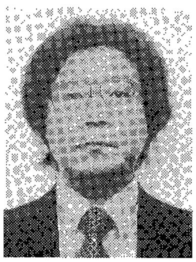

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