PAPER *Special Section on Discrete Mathematics and Its Applications*

On the Complexity of Embedding of Graphs into Grids with Minimum Congestion

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SUMMARY It is known that the problem of determining, given a planar graph G with maximum vertex degree at most 4 and integers m and n, whether G is embeddable in an $m \times n$ grid with unit congestion is NP-hard. In this paper, we show that it is also NP-complete to determine whether G is embeddable in a $k \times n$ grid with unit congestion for any fixed integer $k \geqq 3$. In addition, we show a necessary and sufficient condition for G to be embeddable in a $2 \times \infty$ grid with unit congestion, and show that G satisfying the condition is embeddable in a $2 \times |V(G)|$ grid. Based on the characterization, we suggest a linear time algorithm for recognizing graphs embeddable in a $2 \times \infty$ grid with unit congestion.
key words: *NP-completeness, graph embedding, congestion, grid, VLSI layout*

1. Introduction

The problem of efficiently implementing parallel algorithms on parallel machines and the problem of efficiently laying out VLSI systems onto VLSI chips have been studied as the graph embedding problem, which is to embed a guest graph within a host graph with certain constraints and/or optimization criteria. For the former problem, guest graphs and host graphs represent parallel algorithms and parallel machines, respectively, and the purpose is to minimize communication overhead, such as dilation and/or congestion of the embedding. For the latter problem, a guest graph represents connection requirements of a system and a host graph usually represents a rectangular grid modeling wafer. In VLSI layout, there are various criteria such as wire length, wire congestion, crossing number, and the layout area.

We consider minimal congestion embeddings of graphs into grids. The grids are well-known not only as a model of VLSI chips but also as one of the most popular processor interconnection graphs for parallel machines. It is well-known that the minimal congestion embedding is very important for a grid-connected parallel machine that uses circuit switching for node-to-node communication. In VLSI layout, the minimal congestion embeddings are crucial in the sense that the congestion is a lower bound for the number of layers.

Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We denote by $\Delta(G)$ the maximum degree of a vertex in G. An *embedding* $\langle \phi, \rho \rangle$ of a graph G into a graph H is defined by a one-to-one mapping $\phi : V(G) \rightarrow V(H)$, together with a mapping ρ that maps each edge $(u,v) \in E(G)$ onto a path $\rho(u,v)$ in H that connects $\phi(u)$ and $\phi(v)$. The *congestion* of an edge $e' \in E(H)$ under $\langle \phi, \rho \rangle$ is the number of edges e in G such that $\rho(e)$ contains e'. The congestion of an embedding $\langle \phi, \rho \rangle$ is the maximum congestion of an edge in H. The *one dimensional n-grid* denoted by $M(n)$ is the graph with vertex set $\{0, 1, \ldots, n-1\}$ and edge set $\{(i, i+1) \mid 0 \leqq i \leqq n-2\}$. A Cartesian product $M(n_1) \times M(n_2)$ is called a *two dimensional $n_1 \times n_2$-grid* and denoted by $M(n_1, n_2)$. We define that $n_1 n_2$ is the *area* of $M(n_1, n_2)$. $M(2, n)$ is called an *n-ladder* and denoted by $L(n)$. The embedding of a graph G into a two dimensional grid H is called a *layout* of G into H if it has unit congestion. A layout $\langle \phi, \rho \rangle$ of G into H is said to be *planar* if $\rho(e_1)$ and $\rho(e_2)$ are internally vertex-disjoint for any distinct $e_1, e_2 \in E(G)$.

Formann and Wagner [1] showed that the following problem is NP-complete.

GRAPH LAYOUT I

> **Instance** A planar graph G with $\Delta(G) \leqq 4$ and an integer A.
>
> **Question** Does there exist a layout of G into the grid of area at most A?

Kramer and Leeuwen [3] showed that GRAPH LAYOUT I can be reduced to the following problem:

GRAPH LAYOUT II

> **Instance** A planar graph G with $\Delta(G) \leqq 4$ and integers m, n.
>
> **Question** Does there exist a layout of G into $M(m, n)$?

and thus GRAPH LAYOUT II is NP-hard*.

We consider the following problem which is a variant of GRAPH LAYOUT II:

GRAPH k-LAYOUT

> **Instance** A planar graph G with $\Delta(G) \leqq 4$ and an integer n.

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*[3] claimed that GRAPH LAYOUT II is in NP without proof. However, this is not trivial as mentioned in Sect. 3.2.

Question Does there exist a layout of G into $M(k,n)$?

We prove that the GRAPH k-LAYOUT is NP-complete for any fixed $k \geqq 3$. GRAPH 1-LAYOUT can be trivially solved in polynomial time. Although we do not know the complexity of GRAPH 2-LAYOUT, we consider a closely related problem of laying out a graph into a ladder. We show a necessary and sufficient condition for a graph to be laid out into $L(\infty)$ and show that the graph satisfying the condition can be laid out into $L(|V(G)|)$. Based on the characterization, we suggest a linear time algorithm for deciding if a given graph can be laid out into $L(\infty)$.

The paper is organized as follows. Some definitions are given in Sect. 2. In Sect. 3, we prove the NP-completeness of GRAPH k-LAYOUT for any fixed integer $k \geqq 3$. In Sect. 4, we review the proper-path-width of graphs and show some lemmas used in the following section. In Sect. 5, we give a necessary and sufficient condition for a graph to be laid out into $L(\infty)$. We conclude the paper with some remarks in Sect. 6.

2. Preliminaries

$\Gamma_G(v)$ is the set of edges incident to a vertex v in a graph G. $|\Gamma_G(v)|$ is called the *degree* of v and denoted by $\deg_G(v)$. For $S \subseteqq V(G)$, let $\Gamma_G(S) = \bigcup\{\Gamma_G(v) \mid v \in S\}$. $G[S]$ is the subgraph of G induced by $S \subseteqq V(G)$.

For graphs G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. We write $G \subseteqq H$ if G is a subgraph of H. For an embedding $\varepsilon = \langle \phi, \rho \rangle$ of G into H and $G' \subseteqq G$, let $\varepsilon(G') = \bigcup_{e \in E(G')} \rho(e)$.

Let $M = M(n_1, n_2)$. For a vertex $(i,j) \in V(M)$, let $l_1(i,j) = i$ and $l_2(i,j) = j$. Let $R_i^M = \{(i,j) \in V(M) \mid 0 \leqq j \leqq n_2 - 1\}$ and $C_j^M = \{(i,j) \in V(M) \mid 0 \leqq i \leqq n_1 - 1\}$. Subgraphs $M[R_i^M]$ and $M[C_j^M]$ are called the ith *row* and the jth *column* of M, respectively. For an embedding $\langle \phi, \rho \rangle$ of M and a vertex $(i,j) \in V(M)$, we denote $\phi((i,j))$ simply by $\phi(i,j)$.

3. NP-Completeness of GRAPH k-LAYOUT

We prove the following theorem in this section.

Theorem 1: GRAPH k-LAYOUT is NP-complete for any fixed integer $k \geqq 3$.

We prove in Sect. 3.1 that GRAPH k-LAYOUT ($k \geqq 3$) is NP-hard by constructing a pseudo-polynomial reduction from 3-PARTITION which is well-known to be NP-complete in the strong sense to GRAPH k-LAYOUT. We show that GRAPH k-LAYOUT is in NP in Sect. 3.2.

3.1 NP-Hardness of GRAPH k-LAYOUT

3-PARTITION is defined as follows.

3-PARTITION

Instance A positive integer B, and a set of $3m$ integers $A = \{a_0, a_1, \ldots, a_{3m-1}\}$, such that $B/4 < a_x < B/2$ and $\sum_{x=0}^{3m-1} a_x = mB$.

Question Can A be partitioned into m disjoint sets $A_0, \ldots, A_{m-1}$ such that $\sum_{a \in A_y} a = B$ for $0 \leqq y \leqq m-1$?

For given integers $B, a_0, \ldots, a_{3m-1}$ as an instance of 3-PARTITION, we construct the instance of GRAPH k-LAYOUT as follows:

$$G(A,B) = F(B,m,k) \cup \bigcup_{0 \leqq x \leqq 3m-1} M(a_x),$$

$$n(A,B) = m(B+k+1)+k+1,$$

where $F(B,m,k)$ is the graph obtained from $M(k, n(A,B))$ by removing the vertex $(1,j)$ and joining $(0,j)$ and $(2,j)$ by an edge for each $j = (B+k+1)y+z+k+1$ ($0 \leqq y \leqq m-1, 0 \leqq z \leqq B-1$). Figure 1 shows $F(B,m,k)$. It should be noted that G is well-defined if $k \geqq 3$.

Throughout this subsection, $k \geqq 3$ is a fixed integer. For $0 \leqq y \leqq m$, we define that $J_y^M = \{(B+k+1)y+z \mid 0 \leqq z \leqq k\}$, $J_y^{\overline{M}} = \{(B+k+1)y+z \mid 1 \leqq z \leqq k-1\}$, $M_y = F(B,m,k)[\{(i,j) \mid 0 \leqq i \leqq k-1, j \in J_y^M\}]$, and $\overline{M}_y = F(B,m,k)[\{(i,j) \mid 0 \leqq i \leqq k-1, j \in J_y^{\overline{M}}\}]$. It should be noted that M_y is isomorphic to $M(k,k+1)$ for each $0 \leqq y \leqq m$. Moreover, for $0 \leqq y \leqq m-1$, we define that $J_y^D = \{(B+k+1)y+z+k \mid 0 \leqq z \leqq B+1\}$, and $D_y = F(B,m,k)[\{(i,j) \mid 0 \leqq i \leqq k-1, j \in J_y^D\}]$.

Now we show that A can be partitioned into disjoint sets $A_0, \ldots, A_{m-1}$ such that $\sum_{a \in A_y} a = B$ for $0 \leqq y \leqq m-1$ if and only if there exists an layout of $G(A,B)$ into $H = M(k, n(A,B))$ by a series of lemmas.

Lemma 2: For any layout $\varepsilon = \langle \phi, \rho \rangle$ of $M = M(k, k+$

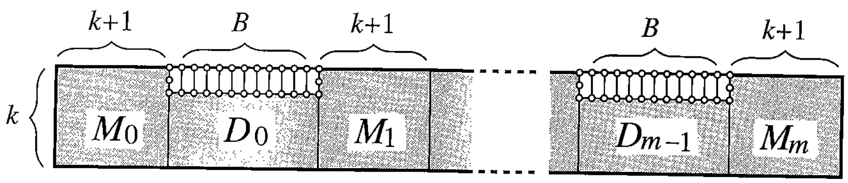

Fig. 1 $F(B,m,k)$. The gray area is grid-connected.

1) into $H = M(k, n(A,B))$,

$$0 \leqq \forall i \leqq k-1\ \exists i' : \quad \varepsilon(M[R_i^M - \{(i,0),(i,k)\}]) \subseteq H[R_{i'}^H], \tag{1}$$

$$1 \leqq \forall j \leqq k-1\ \exists j' : \quad \varepsilon(M[C_j^M]) = H[C_{j'}^H]. \tag{2}$$

Proof: For $0 \leqq i \leqq k-1$ and $0 \leqq j \leqq k$, let $P_i^R = \varepsilon(M[R_i^M])$ and $P_j^C = \varepsilon(M[C_j^M])$. Let

$$q_1 = \min_{0 \leqq j \leqq k} \max_{0 \leqq i \leqq k-1} l_2(\phi(i,j)),$$

$$q_2 = \max_{0 \leqq j \leqq k} \min_{0 \leqq i \leqq k-1} l_2(\phi(i,j)),$$

and

$$j_1 \in \{0 \leqq j \leqq k \mid \max_{0 \leqq i \leqq k-1} l_2(\phi(i,j)) = q_1\},$$

$$j_2 \in \{0 \leqq j \leqq k \mid \min_{0 \leqq i \leqq k-1} l_2(\phi(i,j)) = q_2\}.$$

It follows from the definitions of q_1 and q_2 that

$$0 \leqq \forall j \leqq k \quad \exists v_1 \in C_j^M : \ q_1 \leqq l_2(\phi(v_1)), \text{ and} \quad \exists v_2 \in C_j^M : \ q_2 \geqq l_2(\phi(v_2)). \tag{3}$$

Claim 3: $q_1 < q_2$.

Proof: If $q_1 > q_2$ then it follows from (3) that $P_0^C, \ldots, P_k^C$ are $k+1$ edge-disjoint trails across the columns between the q_1st column and the q_2nd column of H. However, this is impossible since H has just k rows. Thus, we have $q_1 \leqq q_2$.

It remains to show that $q_1 \neq q_2$. We prove this by contradiction. If $q_1 = q_2 = q$ then it follows from (3) that $P_0^C, \ldots, P_k^C$ are $k+1$ edge-disjoint trails across the qth column of H. Thus $0 < q < n(A,B) - 1$, for otherwise, $q = 0$ or $q = n(A,B) - 1$, and we have that $\phi(C_j^M) \cap C_q^H \neq \emptyset$ for every $0 \leqq j \leqq k$, contradicting that ϕ is one-to-one since $|\{C_j^M\}| > |C_q^H|$. We define that

$$E^- = \{((i,q-1),(i,q)) \in E(H) \mid 0 \leqq i \leqq k-1\},$$

$$E^+ = \{((i,q),(i,q+1)) \in E(H) \mid 0 \leqq i \leqq k-1\}.$$

For each $0 \leqq j \leqq k$, if $\phi(C_j^M) \cap C_q^H = \emptyset$ then there exist $v_1, v_2 \in C_j^M$ such that $l_2(\phi(v_2)) < q < l_2(\phi(v_1))$ from (3). Thus, it follows that for any $0 \leqq j \leqq k$,

$$\phi(C_j^M) \cap C_q^H \neq \emptyset \text{ or} \tag{4}$$

$$E(P_j^C) \cap E^- \neq \emptyset \text{ and } E(P_j^C) \cap E^+ \neq \emptyset. \tag{5}$$

Claim 4: For any $0 \leqq j \leqq k$,

$$E(P_j^C) \cap (E^- \cup E^+) \neq \emptyset. \tag{6}$$

Proof: If there exists $0 \leqq j' \leqq k$ such that $E(P_{j'}^C) \cap (E^- \cup E^+) = \emptyset$, then $P_{j'}^C$ is identical with $H[C_q^H]$. This means that a vertex with degree at least 3 in $C_{j'}^M$ is mapped into $\{(i,q) \in V(H) \mid 1 \leqq i \leqq k-2\}$, and that a vertex with degree at least 2 in $C_{j'}^M$ is mapped into $\{(i,q) \in V(H) \mid i = 0 \text{ or } k-1\}$. Thus, both (4) and (5) do not hold for any $j \neq j'$ $(0 \leqq j \leqq k)$, a contradiction. Therefore, (6) holds for any $0 \leqq j \leqq k$.

End of proof of Claim 4

Claim 5: $j_1 \neq j_2$.

Proof: If $j_1 = j_2$, then $\phi(C_{j_1}^M) = C_q^H$ by definition. Thus, for every $j \neq j_1$ $(0 \leqq j \leqq k)$, (5) holds since (4) does not hold. However, since $(0,q) \in \phi(C_{j_1}^M)$ and $\deg_H(0,q) = 3$, P_j^C does not pass through $(0,q)$ for every $j \neq j_1$ $(0 \leqq j \leqq k)$. Thus P_j^C does not pass through $\Gamma_H(0,q)$ for every $j \neq j_1$ $(0 \leqq j \leqq k)$. Since $P_0^C, \ldots, P_k^C$ are edge-disjoint, it follows from (5) that

$$\sum_{0 \leqq j \leqq k-1} |E(P_j^C) \cap E^-| + \sum_{0 \leqq j \leqq k-1} |E(P_j^C) \cap E^+| + |\Gamma_H(0,q) \cap (E^- \cup E^+)| \geqq k + k + 2 = 2k + 2.$$

However, this is a contradiction since the left hand side of the inequality is no more than $|E^- \cup E^+| = 2k$. Therefore, we have $j_1 \neq j_2$. *End of proof of Claim 5*

Let

$$C_1 = \{v \in C_{j_1}^M \mid l_2(\phi(v)) = q\}, \text{ and}$$

$$C_2 = \{v \in C_{j_2}^M \mid l_2(\phi(v)) = q\}.$$

Since

$$\forall v \in C_{j_1}^M - C_1 : l_2(\phi(v)) < q, \text{ and}$$

$$\forall v \in C_{j_2}^M - C_2 : l_2(\phi(v)) > q$$

by definition, it follows that

$$\forall i \in X_1 : E(P_i^R) \cap E^- \neq \emptyset, \tag{7}$$

$$\forall i \in X_2 : E(P_i^R) \cap E^+ \neq \emptyset, \tag{8}$$

where

$$X_1 = \{0 \leqq i \leqq k-1 \mid (i, j_1) \in C_{j_1}^M - C_1\},$$

$$X_2 = \{0 \leqq i \leqq k-1 \mid (i, j_2) \in C_{j_2}^M - C_2\}.$$

Since $P_0^C, \ldots, P_k^C$ and $P_0^R, \ldots, P_{k-1}^R$ are edge-disjoint, we have

$$\sum_{0 \leqq j \leqq k} |E(P_j^C) \cap (E^- \cup E^+)| + \sum_{i \in X_1} |E(P_i^R) \cap E^-| + \sum_{i \in X_2} |E(P_i^R) \cap E^+| \leqq |E^- \cup E^+| = 2k. \tag{9}$$

Since $j_1 \neq j_2$, it follows that $|C_1| + |C_2| = |C_1 \cup C_2| \leqq |C_q^H| = k$. Thus, it follows from (6), (7), and (8) that

$$\begin{aligned}
&(\text{the left hand side of (9)}) \\
&\geqq (k+1) + |X_1| + |X_2| \\
&= (k+1) + (k - |C_1|) + (k - |C_2|) \\
&\geqq (k+1) + 2k - k \\
&\geqq 2k + 1,
\end{aligned}$$

a contradiction. This proves that $q_1 \neq q_2$.

Therefore, we have $q_1 < q_2$.

End of proof of Claim 3

Thus $P_0^R, \ldots, P_{k-1}^R$ are k edge-disjoint trails across the columns between the q_1st column and the q_2nd column of H. Each P_i^R $(0 \leqq i \leqq k-1)$ passes through only edges in one row of $H' = H[\bigcup_{q_1 \leqq j \leqq q_2} C_j^H]$ since H has just k rows. Thus, it follows from (3) that for any $0 \leqq j \leqq k$ $(j \notin \{j_1, j_2\})$ P_j^C passes through only column edges of H'. Therefore, we have $\{j_1, j_2\} = \{0, k\}$, and (1) and (2) hold. □

Throughout this subsection, we assume that $\varepsilon = \langle \phi, \rho \rangle$ is a layout of $F(B, m, k)$ into $H = M(k, n(A, B))$. We may assume without loss of generality that

$$\begin{array}{lll} l_1(\phi(0,1)) & \leqq & l_1(\phi(k-1,k-1)), \text{ and} \\ l_2(\phi(0,1)) & \leqq & l_2(\phi(k-1,k-1)). \end{array} \qquad (10)$$

Lemma 6: For any $0 \leqq y \leqq m$,

$$\begin{aligned} &0 \leqq \forall i \leqq k-1 \ \exists i' : \\ &\quad \varepsilon(F(B,m,k)[\{(i,j) \mid j \in J_y^{\overline{M}}\}]) \subseteqq H[R_{i'}^H], \qquad (11) \\ &\forall j \in J_y^{\overline{M}} \ \exists j' : \\ &\quad \varepsilon(F(B,m,k)[\{(i,j) \mid 0 \leqq i \leqq k-1\}]) \\ &\quad = H[C_{j'}^H]. \qquad (12) \end{aligned}$$

Proof: Immediate from Lemma 2. □

Corollary 7: For any $0 \leqq y \leqq m$ and $e \in E(F(B,m,k)) - E(\overline{M}_y)$, $\rho(e)$ does not pass through an edge of $\varepsilon(\overline{M}_y)$. □

Lemma 8: For any $j \in J_y^{\overline{M}}$ and $j' \in J_{y'}^{\overline{M}}$ $(j < j',$ $0 \leqq y \leqq y' \leqq m)$,

$$l_2(\phi(0,j)) < l_2(\phi(0,j')). \qquad (13)$$

Proof: We first consider the case when $y = 0$. It follows from Lemma 6 and assumption (10) that

$$l_2(\phi(0,1)) < l_2(\phi(0,2)) < \cdots < l_2(\phi(0,k-1)).$$

Thus, (13) holds for any $j, j' \in J_0^{\overline{M}}$ $(j < j')$. Furthermore, for any $j \in J_0^{\overline{M}}$ and $j' \in J_{y'}^{\overline{M}}$ $(j < j', 0 < y' \leqq m)$, (13) follows from Corollary 7.

We next consider the case when $y > 0$. Suppose $j \in J_y^{\overline{M}}$ and $j' \in J_{y'}^{\overline{M}}$ $(j < j', y \leqq y' \leqq m)$. Let $P = \varepsilon(F(B,m,k)[\{(0,l) \mid k-1 \leqq l \leqq j\}])$. Since $l_2(\phi(0,k-1)) < l_2(\phi(0,j'))$, if $l_2(\phi(0,j)) > l_2(\phi(0,j'))$ then P passes through a vertex in $C_{l_2(\phi(0,j'))}^H$. This means that $\varepsilon(F(B,m,k)[\{(i,j') \mid 0 \leqq i \leqq k-1\}]) \neq C_{l_2(\phi(0,j'))}^H$, contradicting to (12).

Therefore, we have $l_2(\phi(0,j)) < l_2(\phi(0,j'))$ for any $j \in J_y^{\overline{M}}$ and $j' \in J_{y'}^{\overline{M}}$ $(j < j', 0 \leqq y \leqq y' \leqq m)$. □

Lemma 9: For any $j \in J_y^{\overline{M}}$ $(0 \leqq y \leqq m)$ and j', j'' such that $0 \leqq j' < j < j'' \leqq n(A,B) - 1$,

$$\begin{aligned} &\max_{0 \leqq i \leqq k-1} l_2(\phi(i,j')) < l_2(\phi(0,j)) \\ &\quad < \min_{0 \leqq i \leqq k-1} l_2(\phi(i,j'')). \qquad (14) \end{aligned}$$

Proof: Immediate from (12), Corollary 7, and Lemma 8. □

Lemma 10: For any $0 \leqq y \leqq m-1$ and any $j, j' \in J_y^D$ $(j < j')$, $l_2(\phi(0,j)) < l_2(\phi(0,j'))$.

Proof: It follows from Lemma 9 that

$$\begin{aligned} &l_2(\phi(0,(B+k+1)y+k-1)) \\ &\quad < l_2(\phi(0,j)) \\ &\quad < l_2(\phi(0,(B+k+1)(y+1)+1)) \qquad (15) \end{aligned}$$

for any $j \in J_y^D$. Fix $j, j' \in J_y^D$ $(j < j')$ and let $q = l_2(\phi(0,j))$. We define that

$$\begin{aligned} E^- &= \{((i,q-1),(i,q)) \in E(H) \mid 0 \leqq i \leqq k-1\}, \\ E^+ &= \{((i,q),(i,q+1)) \in E(H) \mid 0 \leqq i \leqq k-1\}. \end{aligned}$$

For $i \in \{0, 2, \ldots k-1\}$, let $P_i^R = \varepsilon(F(B,m,k)[\{(i,(B+k+1)y+z+k) \mid -1 \leqq z \leqq B+2\}])$. Since $P_3^R, \ldots, P_{k-1}^R$ are edge-disjoint and each P_i^R $(3 \leqq i \leqq k-1)$ contains at least one edge in E^- and at least one edge in E^+ from (15), it follows that

$$\sum_{3 \leqq i \leqq k-1} |E(P_i^R) \cap E^-| \geqq k-3, \qquad (16)$$

$$\sum_{3 \leqq i \leqq k-1} |E(P_i^R) \cap E^+| \geqq k-3. \qquad (17)$$

First assume that $l_2(\phi(0,j)) > l_2(\phi(0,j'))$. It follows from (15) that P_0^R contains at least 3 edges in E^-, and P_2^R contains at least one edge in E^-. Since $P_0^R, P_2^R, \ldots, P_{k-1}^R$ are edge-disjoint, it follows from (16) that

$$\sum_{i \in \{0,2,\ldots,k-1\}} |E(P_i^R) \cap E^-| \geqq 3 + k - 2 = k + 1,$$

which is a contradiction since the left hand side of the inequality is no more than $|E^-| = k$.

Next assume that $l_2(\phi(0,j)) = l_2(\phi(0,j'))$.

Assume that $l_2(\phi(2,j)) = l_2(\phi(2,j')) = q$. Since all the vertices in $U = \{(0,j),(0,j'),(2,j),(2,j')\} \subset V(F(B,m,k))$ have degree at least 3, none of $P_3^R, \ldots, P_{k-1}^R$ passes through a vertex in $\phi(U)$. Thus none of $P_3^R, \ldots, P_{k-1}^R$ passes through an edge in $\Gamma_H(\phi(U))$. Since $|\Gamma_H(\phi(U)) \cap E^-| \geqq 4$ by the assumption that $\phi(U) \subseteqq C_q^H$, it follows from (16) that

$$\begin{aligned} &\sum_{3 \leqq i \leqq k-1} |E(P_i^R) \cap E^-| + |\Gamma_H(\phi(U)) \cap E^-| \\ &\geqq k - 3 + 4 = k + 1. \end{aligned}$$

This is a contradiction since the left hand side of the inequality is no more than $|E^-| = k$.

Thus, we conclude that $l_2(\phi(2,j)) \neq q$ or $l_2(\phi(2,j')) \neq q$. We assume without loss of generality that $l_2(\phi(2,j)) \neq q$ and show a contradiction. For $i \in \{0,2\}$, let $P_i^{R-} = \varepsilon(F(B,m,k)[\{(i,l) \mid (B+k+1)y+k-1 \leq l \leq j\}])$, and $P_i^{R+} = \varepsilon(F(B,m,k)[\{(i,l) \mid j \leq l \leq (B+k+1)(y+1)+1\}])$. Moreover, let $P_j^C = \varepsilon(F(B,m,k)[\{(i,j) \mid 0 \leq i \leq k-1\}])$.

Case 1 $l_2(\phi(2,j)) < q$: Each of P_0^{R-}, P_2^{R+}, and P_j^C contains at least one edge in E^-, and they together with $P_3^R, \ldots, P_{k-1}^R$ are edge-disjoint. Moreover, none of P_0^{R-}, P_2^{R+}, P_j^C, and $P_3^R, \ldots, P_{k-1}^R$ passes through $\phi(0,j')$. Thus none of P_0^{R-}, P_2^{R+}, P_j^C, and $P_3^R, \ldots, P_{k-1}^R$ passes through an edge in $\Gamma_H(\phi(0,j'))$. Thus, it follows from (16) that

$$\sum_{3 \leq i \leq k-1} |E(P_i^R) \cap E^-| + |\Gamma_H(\phi(0,j')) \cap E^-| + |E(P_0^{R-}) \cap E^-| + |E(P_2^{R+}) \cap E^-| + |E(P_j^C) \cap E^-| \geq k-3+4 = k+1.$$

This is a contradiction since the left hand side of the inequality is no more than $|E^-| = k$.

Case 2 $l_2(\phi(2,j)) > q$: Let $P' = \varepsilon(F(B,m,k)[\{(2,j), (2,j'), (0,j')\}])$. Each of P_0^{R+}, P_2^{R-}, P_j^C, and P' contains at least one edge in E^+, and they together with $P_3^R, \ldots, P_{k-1}^R$ are edge-disjoint. Thus, it follows from (16) that

$$\sum_{3 \leq i \leq k-1} |E(P_i^R) \cap E^+| + |E(P_0^{R+}) \cap E^+| + |E(P_2^{R-}) \cap E^+| + |E(P_j^C) \cap E^+| + |E(P') \cap E^+| \geq k-3+4 = k+1.$$

This is again a contradiction since the left hand side of the inequality is no more than $|E^+| = k$.

Therefore, we conclude that $l_2(\phi(0,j)) < l_2(\phi(0,j'))$. □

Lemma 11: For any $0 \leq y \leq m$,

$$\forall j \in J_y^{\overline{M}}: \quad \phi(\{(i,j) \mid 0 \leq i \leq k-1\}) = C_j^H, \tag{18}$$

$$\forall j \in J_y^D: \quad \phi(\{(i,j) \mid 0 \leq i \leq k-1\}) \subset \{(i,l) \in V(H) \mid 0 \leq i \leq k-1, l \in J_y^D\}. \tag{19}$$

Proof: It follows from Lemmas 9 and 10 that $l_2(\phi(0,j)) < l_2(\phi(0,j'))$ for any $0 \leq j < j' \leq n(A,B)-1$. Since H has just $n(A,B)$ columns, we have $l_2(\phi(0,j)) = j$ for any $0 \leq j \leq n(A,B)-1$. Thus, (18) holds by Lemma 6, and (19) holds by (18) and Lemma 9. □

Now we are ready to prove the following.

Lemma 12: GRAPH k-LAYOUT is NP-hard for any fixed integer $k \geq 3$.

Proof: We first assume that A can be partitioned into disjoint sets $A_0, \ldots, A_{m-1}$ such that $\sum_{a \in A_y} a = B$ for $0 \leq y \leq m-1$. We construct a layout $\langle \phi', \rho' \rangle$ of $G(A,B)$ into H as follows: By the definition of $F(B,m,k)$, $F(B,m,k)$ has a planar layout into H such that $\phi'(i,j) = (i,j)$. For each $0 \leq y \leq m-1$, we layout $M(a_x)$ into $H[\{(1,(B+k+1)y+z+k+1) \mid 0 \leq z \leq B-1\}]$ if $a_x \in A_y$. We can construct such layout by the assumption that A can be partitioned into disjoint sets $A_0, \ldots, A_{m-1}$ such that $\sum_{a \in A_y} a = B$ for $0 \leq y \leq m-1$. Thus, we have obtained the desired layout.

Conversely, we assume that there exists a layout $\varepsilon' = \langle \phi', \rho' \rangle$ of $G(A,B)$ into H. For $0 \leq y \leq m-1$, let $U_y = U'_y - \phi'(V(F(B,m,k)))$, where $U'_y = \{(i,j) \in V(H) \mid 0 \leq i \leq k-1, j \in J_y^D\}$. It follows from Lemma 11 that $|U_y| = B$ for $0 \leq y \leq m-1$. Let $U = \bigcup_{0 \leq y \leq m-1} U_y$. Every $M(a_x)$ $(0 \leq x \leq 3m-1)$ is mapped into either U_y or $U - U_y$ by Lemma 11 and the structure of $F(B,m,k)$. This means that A can be partitioned into disjoint sets $A_0, \ldots, A_{m-1}$ such that $\sum_{a \in A_y} a = B$ for $0 \leq y \leq m-1$.

The reduction is pseudo-polynomial since $G(A,B)$ has $kn(A,B) = O(Bm)$ vertices. Thus, GRAPH k-LAYOUT is NP-hard for any fixed integer $k \geq 3$ since 3-PARTITION is NP-complete in the strong sense. □

3.2 GRAPH k-LAYOUT is in NP

In this subsection, we prove that GRAPH k-LAYOUT is in NP. This is not trivial in the sense that every layout of G into H itself may not be a witness of polynomial size if n is much greater than $|V(G)|$. However, the following lemma guarantees that there exists a witness of polynomial size for any instance.

Lemma 13: A graph G which can be laid out into $M(k,n)$ can be laid out into $M(k, 2k|V(G)|)$.

Proof: Let $\varepsilon = \langle \phi, \rho \rangle$ be a layout of G into $H = M(k,n)$. Let $J = \{j \mid \phi(V(G)) \cap C_j^H \neq \emptyset\}$, and we suppose $J = \{j_1, \ldots, j_{|J|}\}$ where $j_1 < \cdots < j_{|J|}$. Obviously, $|J| \leq |V(G)|$. For $1 \leq l \leq |J|-1$, let $E_l = \{((i,j_l),(i,j_l+1)) \in E(H) \mid 0 \leq i \leq k-1\} \bigcup \{((i,j_{l+1}-1),(i,j_{l+1})) \in E(H) \mid 0 \leq i \leq k-1\}$, and $M_l = H[\bigcup_{j_l \leq j \leq j_{l+1}} C_j^H]$. Moreover, let $M_0 = H[\bigcup_{0 \leq j \leq j_1} C_j^H]$, and $M_{|J|} = H[\bigcup_{j_{|J|} \leq j \leq n-1} C_j^H]$.

Suppose that M_l $(1 \leq l \leq |J|-1)$ has more than $2k+1$ columns. If an image of ρ contains an edge in E_l then the image forms one or more subtrail(s) contained in M_l called "net(s)" each of which contains exactly two edges in E_l. Notice that the image contains the even number of edges in E_l since no vertex of G

is mapped by ϕ into $V(M_l) - (C^H_{j_l} \cup C^H_{j_{l+1}})$. Thus, for $1 \leqq l \leqq |J| - 1$, the layout forms a solution of a "channel routing problem" on M_l by considering a vertex in $C^H_{j_l} \cup C^H_{j_{l+1}}$ to be a "terminal" which is connected by a net in M_l. It is known that for a fixed channel length k, if there exists a routing for an instance then there exists a routing with channel width at most $2k-1$ [2]. Thus, we can compact M_l by applying the result so that it has at most $2k+1$ columns.

For M_l ($l \in \{0, |J|\}$), terminals are on only single side of the channel, i.e. C^H_l, and it is easy to see that channel width $\lfloor k/2 \rfloor$ are sufficient for such case. It follows that we can compact M_l so that it has at most $\lfloor k/2 \rfloor + 1$ columns.

Thus, we can obtain a layout of G into $M(k,x)$, where

$$\begin{aligned} x &\leqq (2k-1)(|J|-1) + 2\lfloor k/2 \rfloor + |J| \\ &\leqq 2k|J| - (2k-1) + k \\ &\leqq 2k|V(G)|. \end{aligned}$$

□

Lemma 14: GRAPH k-LAYOUT is in NP.

Proof: Suppose that there exists a layout ε of G into $M(k,n)$. Then $\Delta(G) \leqq 4$ obviously. From Lemma 13, we can assume that n is at most $2k|V(G)|$. Thus, we can check that ε is a layout in $O(|E(M(k,n))||E(G)| + |V(G)|) = O(2kn \cdot 2|V(G)| + |V(G)|) = O(k^2|V(G)|^2)$ time. □

4. Proper-Path-Width

In this section, we review the proper-path-width of a graph, introduced by Takahashi, Ueno, and Kajitani [5], and show some lemmas used in the following section.

Let G be a graph, and let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a sequence of subsets of $V(G)$. The *width* of $\mathcal{X}$ is $\max_{1 \leqq i \leqq r} |X_i| - 1$. $\mathcal{X}$ is called a *proper-path-decomposition* of G if the following conditions are satisfied:

(a) $X_i \not\subseteqq X_j$ $(i \neq j)$;

(b) $\bigcup_{1 \leqq i \leqq r} X_i = V(G)$;

(c) for any $(u,v) \in E(G)$, there exists an i such that $u, v \in X_i$;

(d) for all l, m, and n with $1 \leqq l \leqq m \leqq n \leqq r$, $X_l \cap X_n \subseteqq X_m$;

(e) for all l, m, and n with $1 \leqq l < m < n \leqq r$, $|X_l \cap X_n| \leqq |X_m| - 2$.

The *proper-path-width* of G, denoted by $ppw(G)$, is the minimum width over all proper-path-decompositions of G. A proper-path-decomposition with width k is called a k-proper-path-decomposition. It can be easily seen that $ppw(G) \leqq ppw(H)$ if G is homeomorphic to a subgraph of H. Makedon, Papadimitriou, and Sudborough [4] showed that the *topological band width* of G is identical with the *mixed search number*, called the *node search number* in [4], of G if $\Delta(G) \leqq 3$. They also mentioned a linear time algorithm to determine whether the topological bandwidth of G is at most 2. On the other hand, Takahashi et al. [6] showed that the mixed search number of G is identical with $ppw(G)$. Therefore, we have the following lemma:

Lemma 15: For a graph G with $\Delta(G) \leqq 3$, we can determine whether $ppw(G) \leqq 2$ in $O(|V(G)|)$ time.

A k-proper-path-decomposition $(X_1, X_2, \ldots, X_r)$ is said to be *full* if $|X_i| = k+1$ $(1 \leqq i \leqq r)$ and $|X_j \cap X_{j+1}| = k$ $(1 \leqq i \leqq r-1)$ [6]. The following lemma is shown in [6].

Lemma A: For any graph G with $ppw(G) = k$, there exists a full k-proper-path-decomposition of G.

The following lemma will be used in the next section.

Lemma 16: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a full proper-path-decomposition. For $2 \leqq i \leqq r-1$, there exist a unique $s_i \in X_i - X_{i-1}$ and a unique $t_i \in X_i - X_{i+1}$ $(s_i \neq t_i)$. Moreover, $X_i - \{s_i, t_i\} = X_{i-1} \cap X_{i+1}$.

Proof: It is obvious from the definition (a) that there exist $s_i \in X_i - X_{i-1}$ and $t_i \in X_i - X_{i+1}$ for $2 \leqq i \leqq r-1$. Since $\mathcal{X}$ is full, it follows that $|X_i - X_{i-1}| = |X_i - X_{i+1}| = 1$, so we have $X_i - \{s_i\} \subset X_{i-1}$ and $X_i - \{t_i\} \subset X_{i+1}$. Thus, $X_i - \{s_i, t_i\} \subseteqq X_{i-1} \cap X_{i+1}$. It follows from the definition (e) that $|X_i| - 2 \geqq |X_{i-1} \cap X_{i+1}| \geqq |X_i| - |\{s_i, t_i\}|$. Therefore, we have $s_i \neq t_i$ and $X_i - \{s_i, t_i\} = X_{i-1} \cap X_{i+1}$ for $2 \leqq i \leqq r-1$. □

Corollary 17: Let $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ be a full 2-proper-path-decomposition. For $2 \leqq i \leqq r-1$, there exist a unique $s_i \in X_i - X_{i-1}$, a unique $t_i \in X_i - X_{i+1}$ $(s_i \neq t_i)$, and a unique $v_i \in X_{i-1} \cap X_{i+1}$. □

5. Graph Layout into Ladders

In this section, we show a necessary and sufficient condition for a graph G to be laid out into $L(\infty)$ based on the proper-path-width of G, and show that G satisfying the condition is embeddable into $L(|V(G)|)$. Based on the characterization, we suggest a linear time algorithm for deciding if a given graph can be laid out into $L(\infty)$.

Lemma 18: If a graph G can be laid out into $L(\infty)$, then $\Delta(G) \leqq 3$ and $ppw(G[S]) \leqq 2$, where $S = \{v \in V(G) \mid \deg_G(v) \geqq 2\}$.

Proof: Suppose that there exists a layout $\langle \phi, \rho \rangle$ of G into $L(\infty)$. Then, we have $\Delta(G) \leqq 3$ since $\Delta(L(\infty)) \leqq 3$. Moreover, for $(u,v) \in E(G)$ and $w \in V(G) - \{u,v\}$, $\deg_G(w) \leqq 1$ if $\rho(u,v)$ contains $\phi(w)$. Thus, $\rho(e_1)$ and $\rho(e_2)$ are internally vertex-disjoint for any distinct edges $e_1, e_2 \in E(G[S])$. This means that $G[S]$ is homeomorphic to a subgraph of $L(\infty)$. It is not difficult to see that

$ppw(L(n)) \leqq 2$ for any positive integer n. Therefore, we have $ppw(G[S]) \leqq 2$. □

Lemma 19: For a graph G such that $\Delta(G) \leqq 3$, $|V(G)| \geqq 2$, and $ppw(G) \leqq 2$, there exists a planar layout of G into $L(|V(G)|-1)$.

Proof: We denote $L(|V(G)|-1)$ simply by L. It is easy to see that there exists an desired layout of G into L if $ppw(G) = 1$ or $|V(G)| \leqq 3$. Thus we assume that $ppw(G) = 2$ and $|V(G)| \geqq 4$, and we will construct a desired layout $\varepsilon = \langle \phi, \rho \rangle$.

There exists a full 2-proper-path-decomposition $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of G from the assumption that $ppw(G) = 2$ and Lemma A. It should be noted that $r = |V(G)| - 2 \geqq 2$. The following is an algorithm for laying out G into L.

Phase 1 Denote $s_i \in X_i - X_{i-1}$, $t_i \in X_i - X_{i+1}$, and $v_i = X_{i-1} \cap X_{i+1}$ for $2 \leqq i \leqq r-1$ according to Corollary 17. In addition, let t_1 be a unique element in $X_1 - X_2$, s_r be a unique element in $X_r - X_{r-1}$, $v_r = v_{r-1} (\in X_r)$, and $t_r = s_{r-1}$.

Phase 2 Set $\phi(t_1) = (0,0)$, $\phi(v_2) = (1,1)$, and $\phi(t_2) = (0,1)$. If (t_1, v_2), (t_1, t_2), and (v_2, t_2) are contained in $E(G)$, then set $\rho(t_1, v_2) = L[\{(0,0),(1,0),(1,1)\}]$, $\rho(t_1, t_2) = L[\{(0,0),(0,1)\}]$, and $\rho(v_2, t_2) = L[\{(1,1),(0,1)\}]$.

Phase 3 Execute the following for $i = 2$ to r:

(a) Set $\phi(s_i) = (l_1(\phi(t_i)), i)$. Let

$$P_1 = L[\{(l_1(\phi(t_i)), j) \mid l_2(\phi(t_i)) \leqq j \leqq i\}],$$
$$P_2 = L[\{(l_1(\phi(v_i)), j) \mid l_2(\phi(v_i)) \leqq j \leqq i\}],$$
$$P_3 = L[C_i^L].$$

(b) If $(t_i, s_i) \in E(G)$, then set $\rho(t_i, s_i) = P_1$.

(c) If $(s_i, v_i) \in E(G)$ and no $s_{i'}$ $(i' > i)$ is adjacent to v_i, set $\rho(v_i, s_i) = P_2 \cup P_3$.

(d) If $(s_i, v_i) \in E(G)$ and there exists $s_{i'}$ $(i' > i)$ adjacent to v_i, reset $\phi(v_i) = (l_1(x), i)$ and $\rho(s_i, v_i) = P_3$, where x is the vertex in L into which v_i was mapped before reseting. Moreover, if there exists $y \in V(G) - \{s_i, s_{i'}\}$ adjacent to v_i, then reset $\rho(y, v_i) = P_0 \cup P_2$, where P_0 is the trail in L in which (y, v_i) was mapped before reseting.

Let $Y_i = \bigcup_{1 \leqq j \leqq i} X_j$. We show that ε is the planar layout of G into L by induction on the number of steps in Phase 3. It should be noted that, up to step i in Phase 3, $G[Y_i]$ is laid out into L and that $\phi(v_i)$ may be reset later.

The layout of $G[Y_1]$ defined in Phase 1 is obviously desired one. We assume that ε is the planar layout of $G[Y_{i-1}]$ into $L(|Y_{i-1}|-1)$ for step $i-1$, and show that this is also true for step i. Notice that $|Y_i| = i+2$.

We first show that $\varepsilon(G) \subseteqq L(|Y_i|-1)$. It is easy to see that ϕ is an injection of Y_i since $l_1(\phi(t_i)) \neq l_1(\phi(v_i))$. $\phi(Y_{i-1}) \subseteqq \bigcup_{0 \leqq j \leqq i-1} C_j^L$ by induction hypothesis. After step i, $\phi(s_i) \in C_i^L$ and $\phi(v_i) \in \bigcup_{0 \leqq j \leqq i} C_j^L$ since $t_i \in Y_{i-1}$. This means that $\phi(Y_i) \subset V(L(|Y_i|-1))$. Moreover, the images of ρ defined in step i are contained in $P_1 \cup P_2 \cup P_3$, and $P_1 \cup P_2 \cup P_3 \subseteqq L(|Y_i|-1)$. Thus, we conclude that $\varepsilon(G) \subseteqq L(|Y_i|-1)$.

We next show that ε is the planar layout. Notice that P_1, P_2, and P_3 are internally vertex-disjoint. P_1 and $\rho(e)$ are internally vertex-disjoint for any $e \in E(G[Y_{i-1}])$ since neither vertices nor edges in $\varepsilon(G[Y_{i-1}])$ are contained in $L[\{(l_1(\phi(t_i)), j) \mid j \geqq l_2(\phi(t_i))\}]$ except $\phi(t_i)$. If $(s_i, v_i) \notin E(G)$ then ε is the planar layout since $\varepsilon(G[Y_i]) \subseteqq \varepsilon(G[Y_{i-1}]) \cup P_1$. If $(s_i, v_i) \in E(G)$ then P_2, P_3, and $\rho(e)$ are internally vertex-disjoint for any $e \in E(G[Y_{i-1}])$ since neither vertices nor edges in $\varepsilon(G[Y_{i-1}])$ are contained in $L[\{(l_1(\phi(v_i)), j) \mid j \geqq l_2(\phi(v_i))\}]$ except $\phi(v_i)$. Thus, we conclude that ε is the planar layout. □

Lemma 20: For a graph G such that $\Delta(G) \leqq 3$, $|S| \geqq 2$, and $ppw(G[S]) \leqq 2$, there exists a layout of G into $L(|V(G)|-1)$, where $S = \{v \in V(G) \mid \deg_G(v) \geqq 2\}$.

Proof: It follows from Lemma 19 and the assumption that $\Delta(G[S]) \leqq 3$, $|S| \geqq 2$, and $ppw(G[S]) \leqq 2$ that there exists a planar layout of $G[S]$ into $L(|S|-1)$. Let $v \in V(G) - S$, and let $u \in V(G)$ be a vertex adjacent to v if such u exists. Since $\deg_{G[S]}(u) \leqq 2$, We can map v and (u,v) by adding a new column next to the column containing $\phi(u)$ so that the congestion of the resulting embedding is one. Thus, we can obtain the layout of G into $L(|V(G)|-1)$ since the number of additional columns is at most $|V(G) - S|$. □

We have the following theorem from Lemmas 18 and 20.

Theorem 21: A graph G can be laid out into $L(\infty)$ if and only if $\Delta(G) \leqq 3$ and $ppw(G[S]) \leqq 2$, where $S = \{v \in V(G) \mid \deg_G(v) \geqq 2\}$. □

Based on this theorem, we can obtain a linear time algorithm for deciding if a given graph can be laid out into $L(\infty)$ from Lemma 15.

6. Concluding Remarks

If a full 2-proper-path-decomposition of $G[S]$ is given, the algorithm obtained from the proofs of Lemmas 19 and 20 provides a layout of G into $L(|V(G)|)$ in $O(|V(G)|)$ time. For a graph G with $\Delta(G) \leqq 3$ and $ppw(G) \leqq 2$, we can construct in linear time a full 2-proper-path-decomposition of G based on the result of [4], although the details are omitted here. Therefore, our algorithm can be modified so that it lays out G satisfying the condition of Theorem 21 into $L(|V(G)|)$ in $O(|V(G)|)$ time.

Let $A(G)$ be the minimum area of a ladder into which an N-vertex graph G can be laid out. We can easily modify the algorithm obtained from the proofs of

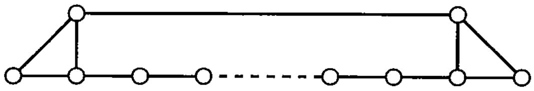

Fig. 2 A graph G with $A(G) = 2(N-2)$.

Lemmas 19 and 20 so that it lays out G into $L(N-2)$ if G has at least 5 vertices with degree at least 2. Thus we have $A(G) \leqq 2(N-2)$. This is the tight bound for $A(G)$ as described in the following corollary.

Corollary 22: If an N-vertex graph G has at least 5 vertices with degree at least 2 then $N \leqq A(G) \leqq 2(N-2)$. Moreover, these are tight bounds, i.e. there exist graphs with $A(G) = N$ and graphs with $A(G) = 2(N-2)$.

Proof: The lower bound is trivial. It is not difficult to see that the graph G shown in Fig. 2 has $A(G) = 2(N-2)$. □

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