Optimal Realization of Hypercubes by Three-Dimensional Space-Invariant Optical Interconnections

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Abstract

It is known that an N-vertex hypercube Q_N can be realized by three-dimensional space-invariant optical interconnections using an optical interconnect module (OIM) with fan-out of size $2\log N - 1$ and two array planes of area $O(N\log^4 N)$. We show that $(8\log N - 12)/5$ and $N(\log N + 1)/2$ are lower bounds for the size of fan-out of OIM and the area of the array plane to realize Q_N, respectively. We also show a realization of Q_N using an OIM with fan-out of size $2\log N$ and two array planes of area $N\log N + N/2$. Our realization is optimal to within a small constant factor.

1 Introduction

Limitations of metal interconnections for high-performance computing systems have been pointed out, and a few optical interconnections have been proposed to overcome these limitations (See, for example, [2]). This paper proposes an optimal realization of hypercubes by free-space optical interconnections.

It is known that free-space interconnections are suitable for chip-to-chip and board-level interconnections. It is also known that space-invariant connections are well matched to the capabilities of optical components, and are easy to implement. This paper considers optimal realizations of interconnection networks based on a model proposed by Louri and Sung [1, 2] for optical interconnections. The model consists of two 2-dimensional arrays on facing planes (array planes) for placing optical components, together with an optical interconnect module (OIM) between them, providing space-invariant connections.

The complexity of the realization based on the model above is measured by the size of fan-out of OIM and the area of the array plane. It is known that an N-vertex hypercube Q_N can be realized using an OIM with fan-out of size $2\log N - 1$ and two array planes of area $O(N\log^4 N)$. We show that $8k/5 - 12/5$ and $(k+1)N/2$ are lower bounds for the size of fan-out of OIM and the area of the array plane to realize a k-regular bipartite graph with N vertices, respectively. In particular, $(8\log N - 12)/5$ and $N(\log N + 1)/2$ are lower bounds for the size of fan-out of OIM and the area of the array plane to realize Q_N, respectively. We also show a realization of Q_N using an OIM with fan-out of size $2\log N$ and two array planes of area $N\log N + N/2$. Our realization is optimal to within a small constant factor.

2 Problem Formulation

2.1 Graph Definitions

Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. A set $I \subseteq V(G)$ is called an independent set of G if no two vertices of I are adjacent in G. G is said to be bipartite if $V(G)$ can be partitioned into two independent sets X and Y. (X, Y) is called a bipartition of G. Let $\deg_G(v)$ denotes the degree of a vertex v in G that is the number of edges of G incident to v. G is said to be k-regular if $\deg_G(v) = k$ for any $v \in V(G)$. A regular graph is one that is k-regular for some k. A set $M \subseteq E(G)$ is called a matching of G if no two edges of M are adjacent in G. A matching M is said to be perfect if for any vertex v, there exists an edge of M incident to v. It is well-known that if G is a k-regular bipartite graph with bipartition (X, Y) then $|X| = |Y|$, and $E(G)$ can be partitioned into k perfect matchings $M_1, M_2, \cdots, M_k$. An edge in M_i is called an i-edge $(i = 1, 2, \cdots, k)$.

2.2 Realization Problem

In this subsection, we define the realization problem for interconnection networks represented by regular bipartite graphs. Let G be a k-regular bipartite graph, and let (X, Y) be a bipartition of G. The vertices and edges of G represent the processors and communication links of an interconnection network, respectively. For each $v \in V(G)$, let $\Gamma(v) = \{v^s, v^1, v^2, \cdots, v^k\}$, where v^s represents the optical source associated with v, and v^i represents an optical detector associated with the i-edge incident to v. This reflects our assumption that a space-division technique is employed for signal separation. Define that $\hat{X} = \bigcup_{x\in X} \Gamma(x)$ and

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$\tilde{Y} = \bigcup_{y \in Y} \Gamma(y)$. Each of $\tilde{X}$ and $\tilde{Y}$ is corresponding to the set of all optical elements to be placed on an array plane. Let $\boldsymbol{Z}$ and $\boldsymbol{N}$ be sets of all integers and all positive integers, repectively. Let $\phi : \tilde{X} \rightarrow \boldsymbol{N}^2$ and $\psi : \tilde{Y} \rightarrow \boldsymbol{N}^2$ be one-to-one mappings. ϕ and ψ represent placements of $\tilde{X}$ and $\tilde{Y}$ on array planes, respectively. Let $\boldsymbol{C} \subseteq \boldsymbol{Z}^2$ be a set of vectors representing a fan-out of OIM that achieves the required connections by space-invariant connections. We assume that $\boldsymbol{C}$ satisfies the following condition: If $\boldsymbol{c} \in \boldsymbol{C}$ then $-\boldsymbol{c} \in \boldsymbol{C}$. This assumption reflects a technical constraint of optics. For any $w, h, i, j \in \boldsymbol{N}$, define that

$$[w \times h]_{i,j} = \{(n, m) \mid w(i-1) < n \leq wi, h(j-1) < m \leq hj\},$$

which is called a module. Notice that $\boldsymbol{N}^2$ is partitioned into modules.

Definition 1 *Given a k-regular bipartite graph G and $w, h \in \boldsymbol{N}$, $\langle \phi, \psi, \boldsymbol{C} \rangle$ is called a realization of G if all of the following conditions are satisfied:*

1. *For any $x \in X[y \in Y]$, $\phi(\Gamma(x))[\psi(\Gamma(y))]$ is contained in a module;*
2. *Any module contains at most one $\phi(\Gamma(x))$ and at most one $\psi(\Gamma(y))$;*
3. *(x, y) is an i-edge if and only if $\psi(y^i) = \phi(x^s) + \boldsymbol{c}$ for some $\boldsymbol{c} \in \boldsymbol{C}$ $(1 \leq i \leq k)$;*
4. *(x, y) is an i-edge if and only if $\phi(x^i) = \psi(y^s) + \boldsymbol{c}'$ for some $\boldsymbol{c}' \in \boldsymbol{C}$ $(1 \leq i \leq k)$.* □

In condition 3[4] above, source $x^s[y^s]$ is said to be connected with detector $y^i[x^i]$ by $\boldsymbol{c}[\boldsymbol{c}']$.

It should be noted that there always exists a realization for any regular bipartite graph if $w \times h$ and $|\boldsymbol{C}|$ are sufficiently large. This can be seen as follows. Let G be a k-regular bipartite graph with N vertices, and (X, Y) be a bipartition of G. Suppose that $w \times h = N$. Define here that $(a, b) \times (c, d) = (ac, bd)$. For any $x \in X[y \in Y]$, let $x^{(i)}[y^{(i)}]$ be a vertex in $Y[X]$ connected with $x[y]$ by an i-edge. Let $\tau_0 : V(G) \rightarrow \{(p, q) \mid p, q \in \boldsymbol{N}, 1 \leq p \leq w, 1 \leq q \leq h\}$ be a one-to-one mapping. There exists such a mapping τ_0 by the assumption that $w \times h = N$. Define one-to-one mappings $\phi_0 : \tilde{X} \rightarrow \boldsymbol{N}^2$ and $\psi_0 : \tilde{Y} \rightarrow \boldsymbol{N}^2$ as follows: For any $x \in X$ and $i (1 \leq i \leq k)$, $\phi_0(x^s) = \tau_0(x) \times (w, h) + \tau_0(x)$ and $\phi_0(x^i) = \tau_0(x) \times (w, h) + \tau_0(x^{(i)})$; For any $y \in Y$ and $i (1 \leq i \leq k)$, $\psi_0(y^s) = \tau_0(y) \times (w, h) + \tau_0(y)$ and $\psi_0(y^i) = \tau_0(y) \times (w, h) + \tau_0(y^{(i)})$. Define that $\boldsymbol{C}_0 = \{(aw, bh) \mid a, b \in \boldsymbol{Z}, |a| \leq w, |b| \leq h\}$. It is easy to see that $\langle \phi_0, \psi_0, \boldsymbol{C}_0 \rangle$ is a realization of G.

The complexity of a realization $\langle \phi, \psi, \boldsymbol{C} \rangle$ of G is measured by $|\boldsymbol{C}|$ and the area of the array plane:

$$A\langle \phi, \psi, \boldsymbol{C} \rangle = \max\{x \mid (x, y) \in \phi(\tilde{X}) \cup \psi(\tilde{Y})\} \times \max\{y \mid (x, y) \in \phi(\tilde{X}) \cup \psi(\tilde{Y})\}.$$

For the realization $\langle \phi_0, \psi_0, \boldsymbol{C}_0 \rangle$ above, $|\boldsymbol{C}_0| = \Theta(N)$ and $A\langle \phi_0, \psi_0, \boldsymbol{C}_0 \rangle = \Theta(N^2)$.

Our problem is to find a realization $\langle \phi, \psi, \boldsymbol{C} \rangle$ for a regular bipartite graph such that both $|\boldsymbol{C}|$ and $A\langle \phi, \psi, \boldsymbol{C} \rangle$ are minimal.

3 Lower Bounds

In this section, we show general lower bounds for the size of fan-out of OIM and the area of the array plane to realize a regular bipartite graph.

Theorem 1 *For any realization $\langle \phi, \psi, \boldsymbol{C} \rangle$ of a k-regular bipartite graph, $|\boldsymbol{C}| \geq 8k/5 - 12/5$.*

Proof Let G be a k-regular bipartite graph with bipartition (X, Y), and $\langle \phi, \psi, \boldsymbol{C} \rangle$ be a realization of G. Without loss of generality, we assume that $(\bigcup_i [w \times h]_{i,1}) \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset$ and $(\bigcup_j [w \times h]_{1,j}) \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset$. Let

$$\begin{aligned}
i(\max) &= \max\{i \mid [w \times h]_{i,1} \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset\}, \\
i(\min) &= \min\{i \mid [w \times h]_{i,1} \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset\}, \\
j(\max) &= \max\{j \mid [w \times h]_{1,j} \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset\}, \\
&\text{and} \\
j(\min) &= \min\{j \mid [w \times h]_{1,j} \cap (\phi(\tilde{X}) \cup \psi(\tilde{Y})) \neq \emptyset\}.
\end{aligned}$$

Define that

$$\begin{aligned}
\boldsymbol{C}_1 &= \{(a, b) \in \boldsymbol{C} \mid ab \geq 0, |b| > h\}, \\
\boldsymbol{C}_2 &= \{(a, b) \in \boldsymbol{C} \mid ab \geq 0, |a| \geq w, |b| \leq h\}, \\
\boldsymbol{C}_3 &= \{(a, b) \in \boldsymbol{C} \mid ab < 0, |b| > h\}, \\
\boldsymbol{C}_4 &= \{(a, b) \in \boldsymbol{C} \mid ab < 0, |a| \geq w, |b| \leq h\}, \text{ and} \\
\boldsymbol{C}_5 &= \{(a, b) \in \boldsymbol{C} \mid |a| < w \text{ and } |b| < h\}.
\end{aligned}$$

Then $(\boldsymbol{C}_1, \boldsymbol{C}_2, \boldsymbol{C}_3, \boldsymbol{C}_4, \boldsymbol{C}_5)$ is a partition of $\boldsymbol{C}$, and we have that $|\boldsymbol{C}| = \sum_{i=1}^{5} |\boldsymbol{C}_i|$. The source in $[w \times h]_{i(\min),1}$ is connected with at most one detector by any pair of vectors $\pm \boldsymbol{c}_1 \in \boldsymbol{C}_1$, at most one detector by any pair of vectors $\pm \boldsymbol{c}_2 \in \boldsymbol{C}_2$, at most one detector by any pair of vectors $\pm \boldsymbol{c}_3 \in \boldsymbol{C}_3$, and at most $|\boldsymbol{C}_4|$ detectors by the vectors in $\boldsymbol{C}_4$. Let S be the number of detectors connected with the source in $[w \times h]_{i(\min),1}$ by the vectors in $\boldsymbol{C}_5$. It is easy to see that $S \leq |\boldsymbol{C}_5|$ if $|\boldsymbol{C}_5| \leq 4$, $S \leq 4$ if $|\boldsymbol{C}_5| = 5$, and $S \leq 5$ if $|\boldsymbol{C}_5| \geq 6$. Since G is k-regular, we conclude that

$$|\boldsymbol{C}_1|/2 + |\boldsymbol{C}_2|/2 + |\boldsymbol{C}_3|/2 + |\boldsymbol{C}_4| + S \geq k. \quad (1)$$

By applying similar arguments to the sources in $[w \times h]_{i(\max),1}$, $[w \times h]_{1,j(\min)}$, and $[w \times h]_{1,j(\max)}$, we have

$$\begin{aligned}
|\boldsymbol{C}_1|/2 + |\boldsymbol{C}_2| + |\boldsymbol{C}_3|/2 + |\boldsymbol{C}_4|/2 + S &\geq k, && (2) \\
|\boldsymbol{C}_1|/2 + |\boldsymbol{C}_2|/2 + |\boldsymbol{C}_3| + |\boldsymbol{C}_4|/2 + S &\geq k, \text{ and} && (3) \\
|\boldsymbol{C}_1| + |\boldsymbol{C}_2|/2 + |\boldsymbol{C}_3|/2 + |\boldsymbol{C}_4|/2 + S &\geq k, && (4)
\end{aligned}$$

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respectively. From inequalities (1), (2), (3) and (4), we obtain that

$$5(|C_1|+|C_2|+|C_3|+|C_4|)/2+4S \geq 4k.$$

Thus

$$|C| = \sum_{i=1}^{5}|C_i| \geq 8k/5+|C_5|-8S/5 \geq 8k/5-12/5.$$

The last inequality follows from the fact that $|C_5|-8S/5$ is minimal if $S=|C_5|=4$. □

Theorem 2 *For any realization $\langle\phi,\psi,C\rangle$ of a k-regular bipartite graph with N vertices, $A\langle\phi,\psi,C\rangle \geq (k+1)N/2$.*

Proof Let G be a k-regular bipartite graph with N vertices, and $\langle\phi,\psi,C\rangle$ be a realization of G. Then,

$$\begin{aligned} & A\langle\phi,\psi,C\rangle \\ &= \max\{x \mid (x,y)\in\phi(\tilde{X})\cup\psi(\tilde{Y})\} \\ &\quad \times \max\{y \mid (x,y)\in\phi(\tilde{X})\cup\psi(\tilde{Y})\} \\ &\geq \max\{x \mid (x,y)\in\phi(\tilde{X})\} \\ &\quad \times \max\{y \mid (x,y)\in\phi(\tilde{X})\} \\ &\geq |\phi(\tilde{X})| \\ &= |\tilde{X}| \\ &= (k+1)N/2. \end{aligned}$$

□

4 Realization of Hypercubes

4.1 Hypercubes

An n-dimensional cube Q_N with $N=2^n$ vertices is defined as follows:

$$\begin{aligned} V(Q_N) &= \{0,1\}^n; \\ E(Q_N) &= \{(u,v) \mid u,v\in V(Q_N), d_H(u,v)=1\}, \end{aligned}$$

where $d_H(u,v)$ denotes the Hamming distance between u and v. An edge is called a dimension i edge if it connects two vertices that differ in the ith bit position ($1\leq i\leq n$). It is easy to see that Q_N is an n-regular bipartite graph. If

$$\begin{aligned} X_Q &= \{x \mid x\in V(Q_N), d_H(x,\mathbf{o}) \text{ is even}\} \text{ and} \\ Y_Q &= \{y \mid y\in V(Q_N), d_H(y,\mathbf{o}) \text{ is odd}\}, \end{aligned}$$

(X_Q,Y_Q) is the bipartition of Q_N. Since the set of dimension i edges is a perfect matching of Q_N, the dimension i edges are regarded as i-edges ($1\leq i\leq n$).

4.2 Lower Bounds

From Theorems 1 and 2, we have the following lower bounds for the realization of hypercubes, since Q_N is a $\log N$-regular bipartite graph with N vertices.

Theorem 3 *For any realization $\langle\phi,\psi,C\rangle$ of Q_N, $|C|\geq(8\log N-12)/5$.* □

Theorem 4 *For any realization $\langle\phi,\psi,C\rangle$ of Q_N, $A\langle\phi,\psi,C\rangle\geq(\log N+1)N/2$.* □

4.3 Realization

Our realization of Q_N is defined as follows.

First, for any $w,h\in\mathbf{N}$, define a mapping $\tau_{w\times h}: V(Q_N)\to\mathbf{N}^2$ as follows: for all $v=v_nv_{n-1}\cdots v_1\in V(Q_N)$,

$$\tau_{w\times h}(v) = \begin{cases} \left(w\sum_{i=1}^{(n-1)/2}2^{i-1}v_{2i-1}, h\sum_{i=1}^{(n-1)/2}2^{i-1}v_{2i}\right) & \text{if } n: \text{odd;} \\ \left(w\sum_{i=1}^{n/2}2^{i-1}v_{2i-1}, h\sum_{i=1}^{(n-2)/2}2^{i-1}v_{2i}\right) & \text{if } n: \text{even.} \end{cases}$$

Next, define mappings $\phi_{w\times h}:\tilde{X}_Q\to\mathbf{N}^2$ and $\psi_{w\times h}:\tilde{Y}_Q\to\mathbf{N}^2$ as follows:

$$\begin{aligned} &\phi_{w\times h}(x^s) = \tau_{w\times h}(x)+(\lceil w/2\rceil,\lceil h/2\rceil); \\ &\phi_{w\times h}(x^i) \\ &= \begin{cases} \tau_{w\times h}(x)+(1,1) & \text{if } i=n, \\ \tau_{w\times h}(x)+(1-x_i)(w',h')+x_i(1,1) \\ \quad +(2x_i-1)(i \bmod w', \lfloor i/w'\rfloor) & \text{if } i<n; \end{cases} \\ &\psi_{w\times h}(y^s) = \tau_{w\times h}(y)+(\lceil w/2\rceil,\lceil h/2\rceil); \\ &\psi_{w\times h}(y^i) \\ &= \begin{cases} \tau_{w\times h}(y)+(1,1) & \text{if } i=n, \\ \tau_{w\times h}(y)+(1-y_i)(w',h')+y_i(1,1) \\ \quad +(2y_i-1)(i \bmod w', \lfloor i/w'\rfloor) & \text{if } i<n; \end{cases} \end{aligned}$$

where $w'=2\lceil w/2\rceil-1$ and $h'=2\lceil h/2\rceil-1$.

Finally, define a set $C_{w\times h}=\{c_0,c_1,\cdots,c_{2n-1}\}$ of vectors as follows:

$$c_i = \begin{cases} (1,1)-(\lceil w/2\rceil,\lceil h/2\rceil) & \text{if } i=0; \\ (2^{(i-1)/2}w,0)-(\lceil w/2\rceil,\lceil h/2\rceil) & \text{if } 1\leq i<n \\ \quad +(1,1)+(i \bmod w', \lfloor i/w'\rfloor) & \quad \text{and } i \text{ is odd;} \\ (0,2^{(i-2)/2}h)-(\lceil w/2\rceil,\lceil h/2\rceil) & \text{if } 1\leq i<n \\ \quad +(1,1)+(i \bmod w', \lfloor i/w'\rfloor) & \quad \text{and } i \text{ is even;} \\ -c_{i-n} & \text{if } n\leq i\leq 2n-1. \end{cases}$$

Lemma 5 *$\langle\phi_{w\times h},\psi_{w\times h},C_{w\times h}\rangle$ is a realization of Q_N if $w'h'=(2\lceil w/2\rceil-1)(2\lceil h/2\rceil-1)\geq 2\log N+1$.*

Proof First, we notice that if $c\in C_{w\times h}$ then $-c\in C_{w\times h}$ by definition.

Let x be a vertex in X_Q and suppose that $\tau_{w\times h}(x)=(aw,bh)$ for some $a,b\in\mathbf{Z}$. Then $\phi_{w\times h}(x^s),\phi_{w\times h}(x^n)\in[w\times h]_{a+1,b+1}$ by definition. We will show that $\phi_{w\times h}(x^i)\in[w\times h]_{a+1,b+1}$ for any i ($1\leq i<n$). Suppose first that $1\leq i<n$ and $x_i=0$. Then

$$\begin{aligned} \phi_{w\times h}(x^i) &= \tau_{w\times h}(x)+(w',h')-(i \bmod w', \lfloor i/w'\rfloor) \\ &= (aw+w'-(i \bmod w'), bh+h'-\lfloor i/w'\rfloor) \end{aligned}$$

by definition. Notice that

$$aw < aw+w'-(i \bmod w') \leq aw+w' \leq w(a+1).$$

From the assumption that $w'h'\geq 2n+1$, we have that $\lfloor i/w'\rfloor \leq \lfloor ih'/(2n+1)\rfloor < \lfloor h'/2\rfloor$. It follows that

$$bh < bh+h'-\lfloor i/w'\rfloor \leq bh+h' \leq h(b+1).$$

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Thus we conclude that $\phi_{w\times h}(x^i) \in [w \times h]_{a+1,b+1}$. Suppose next that $1 \le i < n$ and $x_i = 1$. Then

$$\begin{aligned} \phi_{w\times h}(x^i) &= \tau_{w\times h}(x) + (1,1) + (i \bmod w', \lfloor i/w' \rfloor) \\ &= (aw + 1 + (i \bmod w'), bh + 1 + \lfloor i/w' \rfloor) \end{aligned}$$

by definition. Notice that

$$aw < aw + 1 + (i \bmod w') \le aw + w' \le w(a+1) \text{ and}$$

$$bh < bh+1+\lfloor i/w' \rfloor < bh+1+\lfloor h'/2 \rfloor \le bh+h' \le h(b+1).$$

Thus we conclude that $\phi_{w\times h}(x^i) \in [w \times h]_{a+1,b+1}$. Therefore, we obtain that

$$\phi_{w\times h}(\Gamma(x)) \subseteq [w \times h]_{a+1,b+1}.$$

Similarly, we can show that $\psi_{w\times h}(\Gamma(y))$ is contained in a module for any $y \in Y_Q$. Thus $\langle \phi_{w\times h}, \psi_{w\times h}, \boldsymbol{C}_{w\times h} \rangle$ satisfies condition 1 of Definition 1.

Assume that a module contains both $\phi_{w\times h}(\Gamma(x))$ and $\phi_{w\times h}(\Gamma(x'))$ for some distinct $x, x' \in X_Q$. Since $\phi_{w\times h}(x^s)$ and $\phi_{w\times h}(x'^s)$ are contained in a same module, $\tau_{w\times h}(x) = \tau_{w\times h}(x')$ by the definitions of $\phi_{w\times h}$ and $\tau_{w\times h}$. Thus $x_i = x'_i$ for any i $(1 \le i < n)$ by the definition of $\tau_{w\times h}$. Since $x \ne x'$, we have that $x_n \ne x'_n$, that is, x and x' differ in exactly one bit position, contradicting to the assumption that $x, x' \in X_Q$. Thus any module contains at most one $\phi_{w\times h}(\Gamma(x))$. Similarly, we can show that any module contains at most one $\psi_{w\times h}(\Gamma(y))$. Thus $\langle \phi_{w\times h}, \psi_{w\times h}, \boldsymbol{C}_{w\times h} \rangle$ satisfies condition 2 of Definition 1.

Let x be a vertex in X_Q. Then $\phi_{w\times h}(x^s) \ne \phi_{w\times h}(x^n)$ by definition. We first show that $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^s)$ for any i $(1 \le i < n)$. Assume contrary that $\phi_{w\times h}(x^i) = \phi_{w\times h}(x^s)$ for some i $(1 \le i < n)$. If $x_i = 0$ then $\phi_{w\times h}(x^i) = \tau_{w\times h}(x) + (w', h') - (i \bmod w', \lfloor i/w' \rfloor)$. Since $(i \bmod w', \lfloor i/w' \rfloor) = (w' - \lceil w/2 \rceil, h' - \lceil h/2 \rceil)$, $i = (h' - \lceil h/2 \rceil)w' + w' - \lceil w/2 \rceil = (w'h' - 1)/2 \ge n$, contradicting to the assumption that $i < n$. If $x_i = 1$ then $\phi_{w\times h}(x^i) = \tau_{w\times h}(x) + (1,1) + (i \bmod w', \lfloor i/w' \rfloor)$. Since $(i \bmod w', \lfloor i/w' \rfloor) = (\lceil w/2 \rceil - 1, \lceil h/2 \rceil - 1)$, $i = (\lceil h/2 \rceil - 1)w' + \lceil w/2 \rceil - 1 = (w'h' - 1)/2 \ge n$, contradicting to the assumption that $i < n$. Thus $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^s)$ for any i $(1 \le i < n)$. We next show that $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^n)$ for any i $(1 \le i < n)$. Assume contrary that $\phi_{w\times h}(x^i) = \phi_{w\times h}(x^n)$ for some i $(1 \le i < n)$. If $x_i = 0$ then $\phi_{w\times h}(x^i) = \tau_{w\times h}(x) + (w', h') - (i \bmod w', \lfloor i/w' \rfloor)$. Since $(i \bmod w', \lfloor i/w' \rfloor) = (w' - 1, h' - 1)$, $i = w'h' - 1 \ge 2n$, contradicting to the assumption that $i < n$. If $x_i = 1$ then $\phi_{w\times h}(x^i) = \tau_{w\times h}(x) + (1,1) + (i \bmod w', \lfloor i/w' \rfloor)$. Since $(i \bmod w', \lfloor i/w' \rfloor) = (0,0)$, $i = 0$, contradicting to the assumption that $i \ge 1$. Thus $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^n)$ for any i $(1 \le i < n)$. Finally, we will show that $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^j)$ for any distinct $i, j < n$. Assume contrary that $\phi_{w\times h}(x^i) = \phi_{w\times h}(x^j)$ for some distinct $i, j < n$. If $x_i = x_j$ then $(i \bmod w', \lfloor i/w' \rfloor) = (j \bmod w', \lfloor j/w' \rfloor)$, and so $i = j$, contradicting to the assumption that $i \ne j$. If $x_i \ne x_j$ then $(i \bmod w', \lfloor i/w' \rfloor) + (j \bmod w', \lfloor j/w' \rfloor) = (w' - 1, h' - 1)$. Since $\lfloor (i+j)/w' \rfloor \ge \lfloor i/w' \rfloor + \lfloor j/w' \rfloor$, $i + j \ge (h' - 1)w' + ((w' - 1) \bmod w) = (h' - 1)w' + w' - 1 = w'h' - 1 \ge 2n$, contradicting to the assumption that $i, j < n$. Thus $\phi_{w\times h}(x^i) \ne \phi_{w\times h}(x^j)$ for any distinct $i, j < n$. Therefore, we conclude that $\phi_{w\times h}$ is a one-to-one mapping. Similarly, we can show that $\psi_{w\times h}$ is a one-to-one mapping.

Suppose that (x, y) is an edge such that $x \in X_Q$ and $y \in Y_Q$. Suppose first that (x, y) is an n-edge. Then $\tau_{w\times h}(x) = \tau_{w\times h}(y)$. Thus $\psi_{w\times h}(y^n) - \phi_{w\times h}(x^s) = (1,1) - (\lceil w/2 \rceil, \lceil h/2 \rceil) \in \boldsymbol{C}_{w\times h}$. Suppose next that (x, y) is an i-edge for some i $(1 \le i < n)$. Then

$$\begin{aligned} &\psi_{w\times h}(y^i) - \phi_{w\times h}(x^s) \\ &\quad= \tau_{w\times h}(y) - \tau_{w\times h}(x) + (1 - y_i)(w', h') + y_i(1,1) \\ &\qquad +(2y_i - 1)(i \bmod w', \lfloor i/w' \rfloor) - (\lceil w/2 \rceil, \lceil h/2 \rceil) \\ &\quad= \tau_{w\times h}(y) - \tau_{w\times h}(x) + (-1)^{y_i+1}\{(1,1) \\ &\qquad -(\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)\}. \end{aligned}$$

If i is odd then

$$\begin{aligned} &\psi_{w\times h}(y^i) - \phi_{w\times h}(x^s) \\ &\quad= (y_i - x_i)(2^{(i-1)/2}w, 0) + (-1)^{y_i+1}\{(1,1) \\ &\qquad -(\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)\} \\ &\quad= (-1)^{y_i+1}\{(2^{(i-1)/2}w, 0) + (1,1) \\ &\qquad -(\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)\} \in \boldsymbol{C}_{w\times h}. \end{aligned}$$

If i is even then

$$\begin{aligned} &\psi_{w\times h}(y^i) - \phi_{w\times h}(x^s) \\ &\quad= (y_i - x_i)(0, 2^{(i-2)/2}h) + (-1)^{y_i+1}\{(1,1) \\ &\qquad -(\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)\} \\ &\quad= (-1)^{y_i+1}\{(0, 2^{(i-2)/2}h) + (1,1) \\ &\qquad -(\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)\} \in \boldsymbol{C}_{w\times h}. \end{aligned}$$

Thus $\psi_{w\times h}(y^i) - \phi_{w\times h}(x^s) \in \boldsymbol{C}_{w\times h}$ if (x, y) is an i-edge for some i $(1 \le i \le n)$.

Suppose that x and x' are any distinct vertices in X_Q. We will show that $\phi_{w\times h}(x^s) + \boldsymbol{c} \ne \phi_{w\times h}(x'^s) + \boldsymbol{c}'$ for any $\boldsymbol{c}, \boldsymbol{c}' \in \boldsymbol{C}_{w\times h}$. Assume contrary that $\phi_{w\times h}(x^s) + \boldsymbol{c}_a = \phi_{w\times h}(x'^s) + \boldsymbol{c}_b$ for some $\boldsymbol{c}_a, \boldsymbol{c}_b \in \boldsymbol{C}_{w\times h}$. If $\boldsymbol{c}_a = \boldsymbol{c}_b$ then $\phi_{w\times h}(x^s) = \phi_{w\times h}(x'^s)$. Since $\phi_{w\times h}$ is a one-to-one mapping, it follows that $x^s = x'^s$, contradicting to the assumption that $x \ne x'$. Thus $\boldsymbol{c}_a \ne \boldsymbol{c}_b$. Define here that $(a, b) \bmod (w, h) = (a \bmod w, b \bmod h)$. Then $\boldsymbol{c}_a \bmod (w, h) = \boldsymbol{c}_b \bmod (w, h)$ by the definition of $\phi_{w\times h}$. Let

$$\boldsymbol{C}_a = \begin{array}{l} \{((1,1) - (\lceil w/2 \rceil, \lceil h/2 \rceil) + (i \bmod w', \lfloor i/w' \rfloor)) \\ \bmod (w, h) \mid i \in \boldsymbol{Z}, 0 \le i < n\} \end{array}$$

and

$$\boldsymbol{C}_b = \begin{array}{l} \{(-(1,1) + (\lceil w/2 \rceil, \lceil h/2 \rceil) - (i \bmod w', \lfloor i/w' \rfloor)) \\ \bmod (w, h) \mid i \in \boldsymbol{Z}, 0 \le i < n\}. \end{array}$$

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Then $C_a \cup C_b = \{c \bmod (w,h) \mid c \in C_{w\times h}\}$. Suppose first that $c_a \bmod (w,h), c_b \bmod (w,h) \in C_a$ or $c_a \bmod (w,h), c_b \bmod (w,h) \in C_b$. Without loss of generality, we assume that

$$c_a \bmod (w,h) = \begin{pmatrix} (1,1) - (\lceil w/2\rceil, \lceil h/2\rceil) \\ +(i_a \bmod w', \lfloor i_a/w'\rfloor) \end{pmatrix} \bmod (w,h)$$

and

$$c_b \bmod (w,h) = \begin{pmatrix} (1,1) - (\lceil w/2\rceil, \lceil h/2\rceil) \\ +(i_b \bmod w', \lfloor i_b/w'\rfloor) \end{pmatrix} \bmod (w,h)$$

for some i_a and i_b ($0 \le i_a, i_b < n$). Since $c_a \neq c_b$, $i_a \neq i_b$. Since $c_a \bmod (w,h) = c_b \bmod (w,h)$, and $(i \bmod w') \le w-1$ and $\lfloor i/w'\rfloor < h-1$, we have

$$(i_a \bmod w', \lfloor i_a/w'\rfloor) = (i_b \bmod w', \lfloor i_b/w'\rfloor),$$

and so $i_a = i_b$, a contradiction. Suppose next that $c_a \bmod (w,h) \in C_a$ and $c_b \bmod (w,h) \in C_b$, or $c_a \bmod (w,h) \in C_b$ and $c_b \bmod (w,h) \in C_a$. Without loss of generality, we assume that

$$c_a \bmod (w,h) = \begin{pmatrix} (1,1) - (\lceil w/2\rceil, \lceil h/2\rceil) \\ +(i_a \bmod w', \lfloor i_a/w'\rfloor) \end{pmatrix} \bmod (w,h)$$

and

$$c_b \bmod (w,h) = \begin{pmatrix} -(1,1) + (\lceil w/2\rceil, \lceil h/2\rceil) \\ -(i_b \bmod w', \lfloor i_b/w'\rfloor) \end{pmatrix} \bmod (w,h)$$

for some i_a and i_b ($0 \le i_a, i_b < n$). Since $c_a \bmod (w,h) = c_b \bmod (w,h)$,

$$\begin{aligned} & c_a \bmod (w,h) - c_b \bmod (w,h) \\ = & \begin{pmatrix} 2(1,1) - 2(\lceil w/2\rceil, \lceil h/2\rceil) \\ +(i_a \bmod w', \lfloor i_a/w'\rfloor) \\ +(i_b \bmod w', \lfloor i_b/w'\rfloor) \end{pmatrix} \bmod (w,h) \\ = & (0,0). \end{aligned}$$

Since $-w < -w'+1 \le 2 - 2\lceil w/2\rceil + (i_a \bmod w') + (i_b \bmod w') \le w'-1 < w$ and $-h < -h'+1 \le 2 - 2\lceil h/2\rceil + \lfloor i_a/w'\rfloor + \lfloor i_b/w'\rfloor < h$, we have

$$2 - 2\lceil w/2\rceil + (i_a \bmod w') + (i_b \bmod w') = 0$$

and

$$2 - 2\lceil h/2\rceil + \lfloor i_a/w'\rfloor + \lfloor i_b/w'\rfloor = 0.$$

It follows that

$$(i_a + i_b) \bmod w' = (w'-1) \bmod w$$

and

$$\lfloor (i_a+i_b)/w'\rfloor \ge h'-1.$$

Thus $i_a + i_b \ge (h'-1)w' + ((w'-1) \bmod w) = (h'-1)w' + w' - 1 = h'w' - 1 \ge 2n$, contradicting to the assumption that $i_a, i_b < n$. Thus $\phi_{w\times h}(x^s) + c \neq \phi_{w\times h}(x'^s) + c'$ for any $c, c' \in C_{w\times h}$.

Let y be a vertex in Y_Q and i be an integer such that $1 \le i \le n$. Suppose that (x,y) is not an i-edge. For an i-edge (x',y), there exists $c' \in C_{w\times h}$ such that $\psi_{w\times h}(y^i) = \phi_{w\times h}(x'^s) + c'$. Since $\phi_{w\times h}(x^s) + c \neq \phi_{w\times h}(x'^s) + c'$ for any $c \in C_{w\times h}$, $\psi_{w\times h}(y^i) \neq \phi_{w\times h}(x^s) + c$ for any $c \in C_{w\times h}$. Thus we conclude that $\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle$ satisfies condition 3 of Definition 1. Similarly, we can show that $\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle$ satisfies condition 4 of Definition 1. Therefore, we conclude that $\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle$ is a realization of Q_N. □

Theorem 6
If $wh = 2\log N + 1$ then $\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle$ is a realization of Q_N with $|C_{w\times h}| = 2\log N$ and $A\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle \le N\log N + N/2$.

Proof Since both w and h are odd by the assumption, $w'h' = wh = 2\log N + 1$. Thus $\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle$ is a realization of Q_N by Lemma 5. Moreover, $A\langle \phi_{w\times h}, \psi_{w\times h}, C_{w\times h}\rangle \le (\max\{x \mid (x,y) \in \tau_{w\times h}(V(Q_N))\} + w) \times(\max\{y \mid (x,y) \in \tau_{w\times h}(V(Q_N))\} + h) = whN/2 = N(2\log N + 1)/2$. Finally, $|C_{w\times h}| = 2\log N$ by definition. □

Our realzation of hypercubes can be naturally extended to multi-dimensional tori. The details of the realization of tori together with a unifying approach to realize genaral regular bipartite graphs will appear in a forthcoming paper [3].

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