PAPER

Fault-Tolerant Meshes with Efficient Layouts*

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SUMMARY This paper presents a practical fault-tolerant architecture for mesh parallel machines that has t spare processors and has $2(t+2)$ communication links per processor while tolerating at most $t+1$ processor and link faults. We also show that the architecture presented here can be laid out efficiently in a linear area with wire length at most $O(\sqrt{t})$.

key words: *meshes, fault-tolerant graphs, embeddings, layouts, wire length*

1. Introduction

Many existing parallel machines have a mesh topology that is well suited to many signal processing algorithms. However, even a small number of faulty processors and/or communication links can seriously affect the performance of mesh machines. We show a practical fault-tolerant architecture for meshes in which an optimal number of spare processors and an almost optimal number of communication links are added so that the architecture contains a fault-free mesh even in the presence of a bounded number of faults. We also show an efficient layout of the architecture with almost optimal area and wire length. This research is motivated by a subproject of developing a next generation DSP chip of CAD21 Project at TIT.

Our approach is based on a graph model initiated by Hayes [10]. Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Let $\Delta(G)$ denote the maximum degree of G. For any $S \subseteqq V(G)$, $G - S$ is the graph obtained from G by deleting the vertices of S together with the edges incident to the vertices in S. For any $S \subseteqq E(G)$, $G \setminus S$ is the graph obtained from G by deleting the edges of S. Let t_1 and t_2 be positive integers such that $t_1 \leqq t_2$. A graph G is called a (t_1,t_2)-FT ((t_1,t_2)-fault-tolerant) graph for a graph H if $(G \setminus F_e) - F_v$ contains H as a subgraph for every $F_v \subseteqq V(G)$ and $F_e \subseteqq E(G)$ with $|F_v| \leqq t_1$ and $|F_v| + |F_e| \leqq t_2$. Let m be a positive integer and let $[m] = \{0,1,\ldots,m-1\}$. The 2-dimensional $m \times m$ mesh, denoted by $R(m)$, is defined as follows: $V(R(m)) = [m]^2$; $E(R(m)) = \{([i_1,i_2],[j_1,j_2]) | i_1 = j_1, i_2 - j_2 = \pm 1 \text{ or } i_1 - j_1 = \pm 1, i_2 = j_2\}$. An edge $([i_1,i_2],[j_1,j_2])$ is called a 1-dimensional edge if $i_1 - j_1 = \pm 1$ and $i_2 = j_2$, and a 2-dimensional edge if $i_1 = j_1$ and $i_2 - j_2 = \pm 1$.

A number of fault-tolerant graphs for meshes have been proposed [2]–[6], [8], [9]. Among others, Bruck, Cypher, and Ho [3] showed (t,t)-FT graphs for meshes with t spare vertices and maximum degree at most $2(t+2)$, and that their (t,t)-FT graphs can be embedded with wire length $O(\sqrt{t})$. Their construction is based on circulant graphs. They also showed that their $(1,1)$-FT graphs have maximum degree 4 and can be laid out in a linear area with wire length at most 3.

Here we show $(t,t+1)$-FT graphs for meshes with t spare vertices and maximum degree at most $2(t+2)$, and that our $(t,t+1)$-FT graphs can be laid out in a linear area with wire length $O(\sqrt{t})$. Our construction is also based on circulant graphs. We also show that our $(1,2)$-FT graphs have maximum degree 6 and can be laid out in a linear area with wire length at most 6. It should be noted that our graph can tolerate one more edge fault than the graph proposed in [3], while both graphs need almost same resources, i.e., the number of spare vertices is the same, and the number of spare edges is almost same.

The rest of the paper is organized as follows. We present a $(t,t+1)$-FT graph $G_t(m)$ for a mesh $R(m)$ in Sect. 2. A linear time reconfiguration algorithm for $G_t(m)$ is given in Sect. 3. A layout of $G_t(m)$ with linear area and wire length of $O(\sqrt{t})$ is presented in Sect. 4. Section 5 shows a layout of $G_1(m)$ with linear area and wire length at most 6.

2. $(t,t+1)$-FT Graph $G_t(m)$

In this section, we show a $(t,t+1)$-FT graph $G_t(m)$ for $R(m)$. First of all, we define a graph $G^-(m)$ which will be shown to be a $(0,1)$-FT graph for $R(m)$. $G^-(m)$ is defined as follows: $V(G^-(m)) = [m^2]$; $E(G^-(m)) = \{(i,j) \mid j = (i \pm k) \bmod m^2 \text{ for } k = m \text{ or } m+1\}$. Figure 1 shows $G^-(8)$.

It is known that $G^-(m)$ contains $R(m)$ as a subgraph [3]. Here we show a stronger result as follows.

Lemma 1: $G^-(m)$ is a $(0,1)$-FT graph for $R(m)$.

Proof: We prove that $G^-(m) \setminus \{f_e\}$ contains $R(m)$ as a subgraph for any $f_e \in E(G^-(m))$. Assume without loss of generality that $f_e = (m^2-m-1, 0)$ or $f_e = (m^2 -$

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$m,0)$. Define a mapping $\phi : V(R(m)) \to V(G^-(m))$ such that $\phi[i_1, i_2] = ((i_1 + i_2) \bmod m)m + i_1$. It is easy to see that ϕ is a bijection. If $(i_1 + i_2 + 1) \bmod m > 0$ then $\phi[i_1 + 1, i_2] = ((i_1 + i_2 + 1) \bmod m)m + i_1 + 1 = \phi[i_1, i_2] + m + 1$ and $\phi[i_1, i_2 + 1] = ((i_1 + i_2 + 1) \bmod m)m + i_1 = \phi[i_1, i_2] + m$. If $(i_1 + i_2 + 1) \bmod m = 0$ then $\phi[i_1, i_2] = m^2 - m + i_1$, $\phi[i_1 + 1, i_2] = i_1 + 1 = (\phi[i_1, i_2] + m + 1) \bmod m^2$, and $\phi[i_1, i_2 + 1] = i_1 = (\phi[i_1, i_2] + m) \bmod m^2$. Thus, $G^-(m)$ contains $R(m)$ as a subgraph. Since $\phi^{-1}(0) = [0, 0]$, $\phi^{-1}(m^2 - m - 1) = [m - 1, m - 1]$, and $\phi^{-1}(m^2 - m) = [0, m - 1]$, we conclude that $G^-(m) \setminus \{f_e\}$ contains $R(m)$ as a subgraph. Hence $G^-(m)$ is a $(0, 1)$-FT graph for $R(m)$. □

Now we define a graph $G_t(m)$ which will be shown to be a $(t, t+1)$-FT graph for $R(m)$. $G_t(m)$ is defined as follows: $V(G_t(m)) = [m^2 + t]$; $E(G_t(m)) = \{(i, j) \mid j = (i \pm k) \bmod (m^2 + t)$ for $k = m, m+1, \ldots,$ or $m+t+1\}$. A set $\{(i, j) \mid j = (i \pm k) \bmod (m^2 + t)\}$ is called the edges of offset k. Figure 2 shows $G_1(8)$. It is easy to see the following.

Property 1: $\Delta(G_t(m)) = 2t + 4$ if $t \leqq m^2 - 2m - 3$, and $\Delta(G_t(m)) = t + m^2 - 2m + 1$ otherwise. □

Now we show that $G_t(m)$ is a $(t, t + 1)$-FT graph for $R(m)$. Although the following lemma is an immediate consequence of a result proved by Dutt and Hayes in [7], we give a simple proof since it is needed to prove correctness of our reconfiguration algorithm shown in the next section.

Lemma 2: [7] $G_t(m)$ is a (t, t)-FT graph for $G^-(m)$.

Proof: It is sufficient to show that $G_t(m) - F_v$ contains $G^-(m)$ as a subgraph for every $F_v \subseteqq V(G_t(m))$ with $|F_v| = t$. Define a mapping $\phi_t : V(G^-(m)) \to V(G_t(m) - F_v)$ such that $\phi_t(i)$ is the $(i + 1)$-st smallest fault-free vertex in $G_t(m)$. Clearly, ϕ_t is a bijection.

Now we will prove that $(\phi_t(u), \phi_t(v)) \in E(G_t(m) - F_v)$ for any $(u, v) \in E(G^-(m))$. Assume without loss of generality that $v = (u + m) \bmod m^2$ or $v = (u + m + 1) \bmod m^2$. Let $\phi_t(u) = u + \alpha$ and $\phi_t(v) = v + \beta$. It is easy to see that $0 \leqq \alpha \leqq \beta \leqq t$ if $u \leqq v$.

First, consider the case of $v = (u + m) \bmod m^2$. If $0 \leqq u \leqq m^2 - m - 1$ then $v = u + m$, and so $\phi_t(v) - \phi_t(u) = m + (\beta - \alpha)$. Since $m \leqq m + (\beta - \alpha) \leqq m + t$, $(\phi_t(u), \phi_t(v)) \in E(G_t(m) - F_v)$. If $m^2 - m \leqq u \leqq m^2 - 1$ then $v = u + m - m^2$, and so $(\phi_t(v) - \phi_t(u)) \bmod (m^2 + t) = m + t + (\beta - \alpha)$. Since $m \leqq m + t + (\beta - \alpha) \leqq m + t$, $(\phi_t(u), \phi_t(v)) \in E(G_t(m) - F_v)$.

Next, consider the case of $v = (u + m + 1) \bmod m^2$. If $0 \leqq u \leqq m^2 - m - 2$ then $v = u + m + 1$, and so $\phi_t(v) - \phi_t(u) = m + 1 + (\beta - \alpha)$. Since $m + 1 \leqq m + (\beta - \alpha) \leqq m + t + 1$, $(\phi_t(u), \phi_t(v)) \in E(G_t(m) - F_v)$. If $m^2 - m - 1 \leqq u \leqq m^2 - 1$ then $v = u + m + 1 - m^2$, and so $(\phi_t(v) - \phi_t(u)) \bmod (m^2 + t) = m + t + 1 + (\beta - \alpha)$. Since $m + 1 \leqq m + t + 1 + (\beta - \alpha) \leqq m + t + 1$, $(\phi_t(u), \phi_t(v)) \in E(G_t(m) - F_v)$.

Thus, $G_t(m) - F_v$ contains $G^-(m)$ as a subgraph for every $F_v \subseteqq V(G_t(m))$ with $|F_v| = t$. Hence $G_t(m)$ is a (t, t)-FT graph for $G^-(m)$. □

Theorem 1: $G_t(m)$ is a $(t, t + 1)$-FT graph for $R(m)$.

Proof: Notice that if G is a (s_1, s_2)-FT graph for H and H is a (t_1, t_2)-FT graph for I then G is a $(s_1 + t_1, s_2 + t_2)$-FT graph for I. Thus we obtain the theorem from Lemmas 1 and 2. □

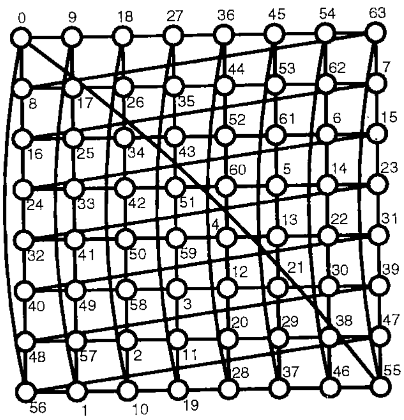
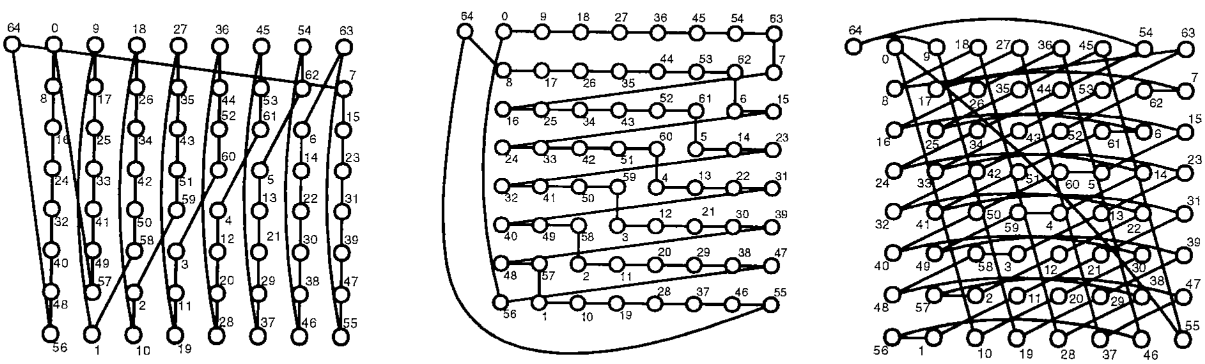


Fig. 1 $G^-(8)$: $(0, 1)$-FT graph for $R(8)$.



(i) The edges of offset 8 (ii) The edges of offset 9 (iii) The edges of offset 10

Fig. 2 $G_1(8)$: $(1, 2)$-FT graph for $R(8)$.

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program Rec_t (input m, f_1, f_2, ..., f_t, (f_x, f_y));
begin
   j := 0; f_{t+1} := ∞;
   for i := 0 to m^2 + t - 1 do
      if (i = f_{j+1}) then
         begin φ_t^{-1}(i) := m^2; j := j + 1 end
      else
         φ_t^{-1}(i) := i - j;
   f_x := φ_t^{-1}(f_x); f_y := φ_t^{-1}(f_y);
   if (f_x ≠ m^2) and (f_y ≠ m^2) then
   begin
      j := 0;
      if ((f_x - f_y) mod m^2 = m or m + 1) then
         j := f_x
      else
         if ((f_y - f_x) mod m^2 = m or m + 1) then
            j := f_y;
      for i := 0 to m^2 + t - 1 do
         if φ_t^{-1}(i) ≠ m^2 then
            φ^{-1}(i) := (φ_t^{-1}(i) - j) mod m^2;
   end ;
   for i := 0 to m^2 + t - 1 do
      if φ^{-1}(i) ≠ m^2 then
         label (i) :=[φ^{-1}(i) mod m,
                       (⌊φ^{-1}(i)/m⌋ - φ^{-1}(i)) mod m]
      else
         label (i) := [∞, ∞];
   return(label );
end
```

Fig. 3 Reconfiguration algorithm Rec_t.

3. Reconfiguration

In this section, we present an efficient algorithm for finding a fault-free $R(m)$ in $G_t(m)$ with t faulty vertices and a faulty edge. If $G_t(m)$ has t' edge faults where $t' \geqq 2$ then select any $t'-1$ of these edges and consider any one of endvertices of each of the edges to be faulty instead of the edge. If $G_t(m)$ has less than $t+1$ faults, then consider fault-free vertices and/or a fault-free edge to be faulty so that $G_t(m)$ has t faulty vertices and a faulty edge. Our reconfiguration algorithm Rec_t is shown in Fig. 3.

It is easy to verify the correctness of Rec_t from the proofs of Lemmas 1 and 2. It is also easy to see that the time complexity of Rec_t is $O(|V(G_t(m))|)$.

4. Layout of $G_t(m)$

4.1 Embedding ψ_t

Let $\mathbb{N} = \{0, 1, \ldots\}$, that is the set of natural numbers. For any graph G, a one-to-one mapping of $V(G)$ into $\mathbb{N}^2$ is called an embedding of G. For any $i = [i_1, i_2]$, $j = [j_1, j_2] \in \mathbb{N}^2$, $dist(i,j) = |i_1 - j_1| + |i_2 - j_2|$. For a graph G and an embedding ϕ of G, let $leng_\phi(G) = \max\{dist(\phi(u), \phi(v)) \mid (u,v) \in E(G)\}$ and let $A_\phi(G) = \max\{x+1 \mid \phi(v) = [x,y],\ v \in V(G)\} \times \max\{y+1 \mid \phi(v) = [x,y],\ v \in V(G)\}$. We define that $leng(G) = \min_\phi leng_\phi(G)$.

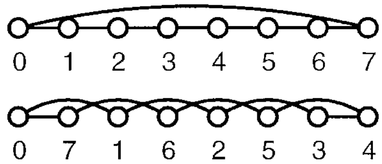


Fig. 4 Interleaved ordering φ_8.

Define a mapping $\varphi_m : [m] \to [m]$ such that $\varphi_m(i) = 2i$ if $0 \leqq i \leqq \lceil m/2 \rceil - 1$, and $\varphi_m(i) = 2(m-i) - 1$ otherwise. Figure 4 shows φ_8. It is easy to see that φ_m is a bijection.

Lemma 3: For any $i \in [m]$, $|\varphi_m(i) - \varphi_m((i+1) \bmod m)| = 1$ if $i = \lceil \frac{m}{2} \rceil - 1$ or $m-1$, and $|\varphi_m(i) - \varphi_m((i+1) \bmod m)| = 2$ otherwise.

Proof: If $0 \leqq i \leqq \lceil \frac{m}{2} \rceil - 2$ then $\varphi_m(i) = 2i$ and $\varphi_m(i+1) = 2(i+1)$, and so $|\varphi_m(i) - \varphi_m(i+1)| = 2$. If $i = \lceil \frac{m}{2} \rceil - 1$ then $\varphi_m(i) = 2i$ and $\varphi_m(i+1) = 2m-2i-3$, and so $\varphi_m(i+1) - \varphi_m(i) = 2m - 4i - 3$. Since $i = \frac{m-2}{2}$ if m is even and $i = \frac{m-1}{2}$ otherwise, we have $|\varphi_m(i) - \varphi_m(i+1)| = 1$. If $\lceil \frac{m}{2} \rceil \leqq i \leqq m-2$ then $\varphi_m(i) = 2m-2i-1$ and $\varphi_m(i+1) = 2m-2i-3$, and so $|\varphi_m(i) - \varphi_m(i+1)| = 2$. If $i = m-1$ then $\varphi_m(m-1) = 1$ and $\varphi_m(0) = 0$, and so $|\varphi_m(i) - \varphi_m((i+1) \bmod m)| = 1$. □

Let $v_1 = \lfloor v/m \rfloor$ and $v_2 = v \bmod m$. Define a one-to-one mapping ψ_t from $V(G_t(m))$ to $\mathbb{N}^2$ as follows:

(i) when $t \leqq m$:

$$\psi_t(v) = \left(\left\lfloor \frac{\varphi_m(v_2)}{\lceil \sqrt{t} \rceil} \right\rfloor, \right.$$
$$\left. (\varphi_m(v_1) + 1)\lceil \sqrt{t} \rceil + \varphi_m(v_2) \bmod \lceil \sqrt{t} \rceil \right)$$

if $v \in [m^2]$, and

$$\psi_t(v) = \left(\left\lfloor \frac{\varphi_t(v_2)}{\lceil \sqrt{t} \rceil} \right\rfloor,\ \varphi_t(v_2) \bmod \lceil \sqrt{t} \rceil \right)$$

otherwise.

(ii) when $m+1 \leqq t < m^2/3$: Define ψ_t as a one-to-one mapping such that $\psi_t(v) \in B_{\varphi_{l_t+1}(l_t - \lfloor v/t \rfloor)}$ for any $v \in V(G_t(m))$, where $l_t = \lfloor (m^2+t-1)/t \rfloor$ and $B_i = [\lceil \sqrt{t} \rceil] \times ([(i+1)\lceil \sqrt{t} \rceil] - [i\lceil \sqrt{t} \rceil])$ for any non-negative integer i.

(iii) when $t \geqq m^2/3$: Define ψ_t as any one-to-one mapping from $V(G_t(m))$ to $[\lceil \sqrt{m^2+t} \rceil]^2$.

Lemma 4:

$$A_{\psi_t}(G_t(m)) \leqq \begin{cases} (m+1)(m+\lceil \sqrt{t} \rceil - 1) & \text{if } t \leqq m, \\ \lceil \sqrt{t} \rceil^2 (l_t + 1) & \text{if } m+1 \leqq t < \frac{m^2}{3}, \\ \lceil \sqrt{m^2+t} \rceil^2 & \text{if } t \geqq \frac{m^2}{3}, \end{cases}$$

where $l_t = \lceil \frac{m^2+t-1}{t} \rceil$.

Proof: If $t \leqq m$ then the maxima of the first and second coordinates of $\psi_t(v)$ over all $v \in V(G_t(m))$ is

$\lfloor \frac{m-1}{\lceil\sqrt{t}\rceil} \rfloor \leqq \frac{m-1}{\lceil\sqrt{t}\rceil}$ and $(m+t)\lceil\sqrt{t}\rceil - 1$, respectively. If $m+1 \leqq t < \frac{m^2}{3}$ then the maxima of the first and second coordinates of $\psi_t(v)$ over all $v \in V(G_t(m))$ is $\lceil\sqrt{t}\rceil - 1$ and $(l_t+1)\lceil\sqrt{t}\rceil - 1$, respectively. If $t \geqq \frac{m^2}{3}$ then $\psi_t(v) \in [\lceil\sqrt{m^2+t}\rceil]$ for any $v \in V(G_t(m))$. □

4.2 Wire Length of ψ_t

Lemma 5: Let t be a positive integer and let a, b, and c be non-negative integers such that $|a-b| \leqq 2t+c$. Then,

$$\left|\left\lfloor \frac{a}{\lceil\sqrt{t}\rceil} \right\rfloor - \left\lfloor \frac{b}{\lceil\sqrt{t}\rceil} \right\rfloor\right| \leqq \begin{cases} 2\lceil\sqrt{t}\rceil + c & \text{if } b > 0, \\ \lfloor 2\sqrt{t} \rfloor + c & \text{if } b = 0. \end{cases}$$

Proof: It is easy to see the case when $b=0$. We will consider the case when $b>0$. Assume without loss of generality that $a \geqq b$. Then,

$$\begin{aligned} \left\lfloor \frac{a}{\lceil\sqrt{t}\rceil} \right\rfloor &\leqq \left\lfloor \frac{b+2t+c}{\lceil\sqrt{t}\rceil} \right\rfloor \\ &\leqq \left\lfloor \frac{b+2\lceil\sqrt{t}\rceil^2}{\lceil\sqrt{t}\rceil} \right\rfloor + c \\ &\leqq \left\lfloor \frac{b}{\lceil\sqrt{t}\rceil} \right\rfloor + 2\lceil\sqrt{t}\rceil + c. \end{aligned}$$

Hence the lemma is true. □

Theorem 2: $leng_{\psi_t}(G_t(m)) \leqq 5\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor$ if $t \leqq m-1$.

Proof: We will show that $dist(\psi_t(u), \psi_t(v)) \leqq 5\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor$ for any $(u,v) \in E(G_t(m))$. Assume without loss of generality that $v = (u+k) \bmod (m^2+t)$ for some $k = m, m+1, \ldots,$ or $m+t+1$. Let $u_1 = \lfloor u/m \rfloor$, $u_2 = u \bmod m$, $v_1 = \lfloor v/m \rfloor$, and $v_2 = v \bmod m$. Note that $u_2 \in [m]$ if $u_1 \in [m]$, and $u_2 \in [t]$ otherwise. So $m \leqq u_2 + k \leqq 2m + t$ if $u_1 \in [m]$, and $m \leqq u_2 + k \leqq m + 2t$ otherwise.

Case 1 $u_1 \leqq m-2$ and $u_2 + k \leqq 2m-1$: Since $(u_1+1)m \leqq u+k \leqq (u_1+2)m - 1$, $v_1 = u_1 + 1$ and $v_2 = u_2 + k - m$. Since $|\varphi_m(v_1) - \varphi_m(u_1)| \leqq 2$ and $|\varphi_m(v_2) - \varphi_m(u_2)| \leqq 2(k-m) \leqq 2(t+1)$ by Lemma 3, we have $dist(\psi_t(u), \psi_t(v)) \leqq 3\lceil\sqrt{t}\rceil - 1 + 2\lceil\sqrt{t}\rceil + 2 \leqq 5\lceil\sqrt{t}\rceil + 1$ by Lemma 5.

Case 2 $u_1 \leqq m-3$ and $u_2 + k \geqq 2m$: Since $(u_1+2)m \leqq u+k \leqq (u_1+2)m + t$, $v_1 = u_1 + 2$ and $v_2 = u_2 + k - 2m$. Since $\varphi_m(u_2) \leqq 2(m-u_2) - 1 \leqq 2(k-m) - 1 \leqq 2t+1$, we have $\psi_t(u) \in [\lfloor 2\sqrt{t} \rfloor + 2] \times ([(\varphi_m(u_1)+2)\lceil\sqrt{t}\rceil] - [(\varphi_m(u_1)+1)\lceil\sqrt{t}\rceil])$. Since $\varphi_m(v_2) \leqq 2(u_2+k-2m) \leqq 2t$, we have $\psi_t(v) \in [\lfloor 2\sqrt{t} \rfloor + 1] \times ([(\varphi_m(u_1+2)+2)\lceil\sqrt{t}\rceil] - [(\varphi_m(u_1+2)+1)\lceil\sqrt{t}\rceil])$. Thus, $dist(\psi_t(u), \psi_t(v)) \leqq 5\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor$.

Case 3 $u_1 = m-2$ and $2m \leqq u_2 + k \leqq 2m + t - 1$: Since $m^2 \leqq u+k \leqq m^2 + t - 1$, $\psi_t(v) \in [\lceil\sqrt{t}\rceil]^2$. Since $\varphi_m(u_2) \leqq 2(m-u_2) - 1 \leqq 2(k-m) - 1 \leqq 2t+1$, we have $\psi_t(u) \in [\lfloor 2\sqrt{t} \rfloor + 2] \times [5\lceil\sqrt{t}\rceil]$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 5\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor$.

Case 4 $u_1 = m-2$ and $u_2 + k = 2m + t$: Since $u_2 = m-1$ and $k = m+t+1$, $u = m^2 - m - 1$ and $v = 0$. Thus $dist(\psi_t(u), \psi_t(v)) \leqq 3\lceil\sqrt{t}\rceil + 1$.

Case 5 $u_1 = m-1$ and $u_2 + k \leqq m + t - 1$: Since $m^2 \leqq u+k \leqq m^2 + t - 1$, $\psi_t(v) \in [\lceil\sqrt{t}\rceil]^2$. Since $\varphi_m(u_2) \leqq 2u_2 \leqq 2(m+t-1-k) \leqq 2(t-1)$, we have $\psi_t(u) \in [\lfloor 2\sqrt{t} \rfloor + 1] \times [3\lceil\sqrt{t}\rceil]$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 3\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor - 1$.

Case 6 $u_1 = m-1$ and $m+t \leqq u_2 + k \leqq 2m + t - 1$: Since $m^2 + t \leqq u+k \leqq m^2 + m + t - 1$, $0 \leqq v \leqq m-1$, and so $v_1 = 0$ and $v_2 = u_2 + k - m - t$. Since $|\varphi_m(m-1) - \varphi_m(0)| = 1$ and $|\varphi_m(v_2) - \varphi_m(u_2)| \leqq 2|k-m-t| \leqq 2t$, we have $dist(\psi_t(u), \psi_t(v)) \leqq 2\lceil\sqrt{t}\rceil - 1 + 2\lceil\sqrt{t}\rceil = 4\lceil\sqrt{t}\rceil - 1$.

Case 7 $u_1 = m-1$ and $u_2 + k = 2m + t$: Since $u_2 = m-1$ and $k = m+t+1$, $u = m^2 - 1$ and $v = m$. Thus, $dist(\psi_t(u), \psi_t(v)) \leqq \lceil\sqrt{t}\rceil + 1$.

Case 8 $u_1 = m$ and $u_2 + k \leqq m + t - 1$: Since $m^2 + m \leqq u+k \leqq m^2 + m + t - 1$, $m - t \leqq v \leqq m-1$, and so $v_1 = 0$ and $v_2 = u_2 + k - t$. Since $\varphi_m(v_2) \leqq 2(m+t-u_2-k) \leqq 2t$, we have $\psi_t(u), \psi_t(v) \in [\lfloor 2\sqrt{t} \rfloor + 1] \times [2\lceil\sqrt{t}\rceil]$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 2\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor - 1$.

Case 9 $u_1 = m$ and $u_2 + k \geqq m + t$: Since $m^2 + m + t \leqq u+k \leqq m^2 + m + 2t$, $m \leqq v \leqq m+t$, and so $v_1 = 1$ and $v_2 = u_2 + k - m - t$. Since $\varphi_m(v_2) \leqq 2(u_2 + k - m - t) \leqq 2t$, we have $\psi_t(u), \psi_t(v) \in [\lfloor 2\sqrt{t} \rfloor + 1] \times [4\lceil\sqrt{t}\rceil]$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 4\lceil\sqrt{t}\rceil + \lfloor 2\sqrt{t} \rfloor - 1$. □

Theorem 3: $leng_{\psi_m}(G_m(m)) \leqq 6\lceil\sqrt{m}\rceil + 1$.

Proof: We will show that $dist(\psi_m(u), \psi_m(v)) \leqq 6\sqrt{m} + 1$ for any $(u,v) \in E(G_m(m))$. Assume without loss of generality that $v = (u+k) \bmod (m^2+m)$ for some $k = m, m+1, \ldots, 2m+1$. Let $u_1 = \lfloor u/m \rfloor$, $u_2 = u \bmod m$, $v_1 = \lfloor v/m \rfloor$, and $v_2 = v \bmod m$. Note that $m \leqq u_2 + k \leqq 3m$. If $u_2 + k \neq 3m$,(that is if $u_2 \neq m-1$ or $k \neq 2m+1$,) then $v_1 = (u_1+1) \bmod (m+1)$ or $(u_1+2) \bmod (m+1)$. Thus, $dist(\psi_m(u), \psi_m(v)) \leqq 6\lceil\sqrt{m}\rceil - 2$. If $u_2 + k = 3m$,(that is if $u_2 = m-1$ and $k = 2m+1$,) then $v_1 = (u_1+3) \bmod (m+1)$ and $v_2 = 0$. Thus, $dist(\psi_m(u), \psi_m(v)) \leqq 6\lceil\sqrt{m}\rceil + 1$. □

Theorem 4: $leng_{\psi_t}(G_t(m)) \leqq 6\lceil\sqrt{t}\rceil - 2$ if $m+1 \leqq t < m^2/3$.

Proof: We will show that $dist(\psi_t(u), \psi_t(v)) \leqq 6\lceil\sqrt{t}\rceil - 2$ for any $(u,v) \in E(G_t(m))$. Assume without loss of generality that $v = (u+k) \bmod (m^2+t)$ for some $k = m, m+1, \ldots,$ or $m+t+1$. Let $u_3 = \lfloor u/t \rfloor$ and $u_4 = u \bmod t$. Note that $m \leqq u_4 + k \leqq m + 2t \leqq 3t - 1$. If $u_3 \leqq l_t - 3$ then $\psi_t(u) \in B_{\varphi_{l_t+1}(l_t-v_3)}$ and $\psi_t(v) \in B_{\varphi_{l_t+1}(l_t-v_3)} \cup B_{\varphi_{l_t+1}(l_t-v_3-1)} \cup B_{\varphi_{l_t+1}(l_t-v_3-2)}$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 6\lceil\sqrt{t}\rceil - 2$. If $u_3 \geqq l_t - 2$ then $\psi_t(u) \in B_0 \cup B_2 \cup B_4$. Since $(l_t - 2)t + m \leqq u + k \leqq m^2 + m + 2t$, $\psi_t(v) \in \bigcup_{i=0}^{4} B_i$. Hence $dist(\psi_t(u), \psi_t(v)) \leqq 6\lceil\sqrt{t}\rceil - 2$. □

Theorem 5: $leng_{\psi_t}(G_t(m)) \leqq 4\lceil\sqrt{t}\rceil - 2$ if $t \geqq m^2/3$.

Proof: For any $(u,v) \in E(G_t(m))$, $dist(\psi_t(u), \psi_t(v)) \leqq 2\lceil\sqrt{m^2+t}\rceil - 2 \leqq 4\lceil\sqrt{t}\rceil - 2$. Hence $leng_{\psi_t}(G_t(m)) \leqq 4\lceil\sqrt{t}\rceil - 2$. □

By summarizing Theorems 2–5, we have the following theorem.

Theorem 6: $leng_{\psi_t}(G_t(m)) = O(\sqrt{t})$. □

4.3 Lower Bound for Wire Length

Lemma 6: For any $x \in \mathbb{N}^2$, $|\{y \in \mathbb{N}^2 | \; dist(x,y) \leqq l\}| \leqq 2l^2 + 2l + 1$.

Proof: For any $x \in \mathbb{N}^2$ and any $l > 0$, $|\{y \in \mathbb{Z}^2 | \; dist(x,y) = l\}| \leqq 4l$, where $\mathbb{Z}$ denotes the integer set. Thus, the number of $y \in \mathbb{N}^2$ such that $dist(x,y) \leqq l$ is at most

$$1 + \sum_{i=1}^{l} 4i = 2l^2 + 2l + 1.$$

□

Lemma 7: $leng(G) \geqq \frac{\sqrt{2 \cdot \Delta(G)+1}-1}{2}$.

Proof: Let l be the integer such that $\frac{\sqrt{2 \cdot \Delta(G)+1}-1}{2} - 1 \leqq l < \frac{\sqrt{2 \cdot \Delta(G)+1}-1}{2}$. Consider any embedding η of G and some $u \in V(G)$ whose degree is $\Delta(G)$. Since there exists at most $2l^2 + 2l + 1 < \Delta(G) + 1$ of $v \in V(G)$ such that $dist(\eta(u), \eta(v)) \leqq l$ by Lemma 6, there exists some $w \in V(G)$ such that $(u,w) \in E(G)$ and $dist(\eta(u), \eta(w)) \geqq l + 1 \geqq \frac{\sqrt{2 \cdot \Delta(G)+1}-1}{2}$. Hence $leng(G) \geqq \frac{\sqrt{2 \cdot \Delta(G)+1}-1}{2}$. □

By Property 1 and Lemma 7, we obtain the following theorem.

Theorem 7:

$$leng(G_t(m)) \geqq \begin{cases} \dfrac{\sqrt{4t+9}-1}{2} & \text{if } t < m^2 - 2m - 2, \\ \dfrac{\sqrt{2t+2m^2-4m+3}-1}{2} & \text{otherwise.} \end{cases}$$

□

From Theorems 6 and 7, we conclude that the wire length of our embedding is optimal to within a small constant factor of 7.

4.4 Layout of $G_t(m)$

We need a few preliminaries on digraphs. Let D be a digraph and let $V(D)$ and $A(D)$ be the vertex set and arc set of D, respectively. An arc from a vertex u to a vertex v is denoted by $(u,v)_d$. For any $v \in V(D)$, Let $\Gamma_D^-(v) = \{u \in V(D) | \; (u,v)_d \in A(D)\}$ and $\Gamma_D^+(v) = \{u \in V(D) | \; (v,u)_d \in A(D)\}$, and let $d_D^-(v) = |\Gamma_D^-(v)|$ and $d_D^+(v) = |\Gamma_D^+(v)|$. A digraph obtained from a graph G by orienting each edge is called an orientation of G. The following lemma about the orientation of a graph is well-known. (See [1], for example.)

Lemma 8: A graph G has an orientation D such that $|d_D^-(v) - d_D^+(v)| \leqq 1$ for all $v \in V(D)$. □

Lemma 9: Let l and γ be positive integers and let $\beta = 2(l+1)\gamma + 1$. Let a, b, and c be integers such that $a \geqq 1$, $|b| \leqq 2l+1$, and $|c| \leqq \gamma$. Then $a\beta + b\gamma + c > 0$.

Proof: $a\beta + b\gamma + c > 2(l+1)\gamma + b\gamma + c \geqq |b|\gamma + |c| + b\gamma + c \geqq 0$. □

Lemma 10: Let l and γ be positive integers and let $\beta = 2(l+1)\gamma + 1$. If $a\beta + b\gamma + c = 0$ for some integers a, b, and c such that $|b| \leqq 2l+1$ and $|c| \leqq \gamma$, then $(a,b,c) = (0,0,0)$, $(0,1,-\gamma)$, or $(0,-1,\gamma)$.

Proof: If $a\beta + b\gamma + c = 0$ for some integers a, b, and c such that $|b| \leqq 2l+1$ and $|c| \leqq \gamma$, then $a = 0$ by Lemma 9, and so $b\gamma + c = 0$. If $|b| \geqq 2$ then

$$2\gamma \leqq |b\gamma| = |c| \leqq \gamma < 2\gamma,$$

which is a contradiction. Thus, $b = 0$, 1, or -1, and hence $(a,b,c) = (0,0,0)$, $(0,1,-\gamma)$, or $(0,-1,\gamma)$. □

Now, we are ready to prove the following key lemma.

Lemma 11: Let G be a graph and ν be an embedding of G such that $leng_\nu(G) \leqq l$. Then G has a layout with area $\{2(l+1)\lceil\Delta(G)/2\rceil + 1\}^2 \times A_\nu(G)$.

Proof: Let D be an orientation of G such that $|d_D^-(v) - d_D^+(v)| \leqq 1$ for all $v \in V(D)$. We consider a layout of D. For each vertex v of D, assign a $\gamma \times \gamma$ square with four corners $[v_1\beta, \; v_2\beta]$, $[v_1\beta+\gamma, \; v_2\beta]$, $[v_1\beta+\gamma, \; v_2\beta+\gamma]$, and $[v_1\beta, \; v_2\beta+\gamma]$, where $\gamma = \lceil\Delta(G)/2\rceil$, $\beta = 2(l+1)\gamma+1$, and $\nu(v) = [v_1, v_2]$. For any $v \in V(D)$, define $f_v^- : \Gamma_D^-(v) \to [\gamma]$ and $f_v^+ : \Gamma_D^+(v) \to [\gamma]$ to be any one-to-one mappings. f_v^- and f_v^+ always exist by the definition of D. Then, for any $(u,v)_d \in A(D)$, define the route $P(u,v)$ between two $\gamma \times \gamma$ squares corresponding to u and v as follows. We use the so called two layer routing, in which vertical and horizontal segments are routed on the first and second layers, respectively.

$$\begin{aligned} P(u,v): \; & [u_1\beta + f_u^+(v), \; u_2\beta + \gamma] \\ & \xrightarrow{1} [u_1\beta + f_u^+(v), \; u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1] \\ & \xrightarrow{2} [v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1, \\ & \qquad\qquad u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1] \\ & \xrightarrow{1} [v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1, \; v_2\beta + f_v^-(u)] \\ & \xrightarrow{2} [v_1\beta + \gamma, \; v_2\beta + f_v^-(u)], \end{aligned}$$

where $\nu(u) = [u_1, u_2]$, $h_l(x) = x \bmod (2l+1) + 1$, and $\xrightarrow{i}$ denotes the connection by a segment in layer i for $i = 1, 2$. It is easy to see that the area of this layout is $\beta^2 A_\nu(G) = \{2(l+1)\lceil\Delta(G)/2\rceil + 1\}^2 \times A_\nu(G)$.

It remains to show that this is really a layout, i.e., to show that if $(u,v)_d \neq (w,z)_d$ then a segment in $P(u,v)$ has no intersection with a segment in $P(w,z)$. Assume contrary that a segment in $P(u,v)$ and a segment in $P(w,z)$ intersect. Assume without loss of generality that those segments are on layer 2. Let $\nu(w) = [w_1, w_2]$ and $\nu(z) = [z_1, z_2]$. Let seg_1 and seg_2 denote segment:

$$[u_1\beta + f_u^+(v), \; u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1]$$

$$\xrightarrow{2} [v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1,\ u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1]$$

and segment:

$$[v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1, v_2\beta + f_v^-(u)]$$
$$\xrightarrow{2} [v_1\beta + \gamma, v_2\beta + f_v^-(u)]$$

in $P(u, v)$, respectively, and let seg_3 and seg_4 denote segment:

$$[w_1\beta + f_w^+(z),\ w_2\beta + h_l(w_1)\gamma + f_w^+(z) + 1]$$
$$\xrightarrow{2} [z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1,\ w_2\beta + h_l(w_1)\gamma + f_w^+(z) + 1]$$

and segment:

$$[z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1, z_2\beta + f_z^-(w)]$$
$$\xrightarrow{2} [z_1\beta + \gamma, z_2\beta + f_z^-(w)]$$

in $P(w, z)$, respectively.

Claim 1: seg_1 and seg_4 are disjoint.

Proof of Claim 1: If seg_1 and seg_4 intersect, we have

$$u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1 = z_2\beta + f_z^-(w),$$

that is,

$$(u_2 - z_2)\beta + h_l(u_1)\gamma + \{f_u^+(v) - f_z^-(w) + 1\} = 0.$$

Since $1 \leqq h_l(u_1) \leqq 2l + 1$ and $-\gamma + 2 \leqq f_u^+(v) - f_z^-(w) + 1 \leqq \gamma$, we have $u_2 - z_2 = 0$, $h_l(u_1) = 1$, and $f_u^+(v) - f_z^-(w) + 1 = -\gamma$ by Lemma 10, which is a constradiction. □

Similarly, we can prove the following claim.

Claim 2: seg_2 and seg_3 are disjoint. □

Claim 3: seg_1 and seg_3 are disjoint.

Proof of Claim 3: Assume that seg_1 and seg_3 intersect. Then, $u_2\beta + h_l(u_1)\gamma + f_u^+(v) + 1 = w_2\beta + h_l(w_1)\gamma + f_w^+(z) + 1$, that is, $(u_2 - w_2)\beta + \{h_l(u_1) - h_l(w_1)\}\gamma + \{f_u^+(v) - f_w^+(z)\} = 0$. Since $|h_l(u_1) - h_l(w_1)| \leqq 2l$ and $|f_u^+(v) - f_w^+(z)| \leqq \gamma - 1$, we have that $u_2 = w_2$, $h_l(u_1) = h_l(w_1)$, and $f_u^+(v) = f_w^+(z)$ by Lemma 10. If $u_1 = w_1$ then $u = w$. Since $f_u^+(v) = f_w^+(z)$, we have that $(u, v)_d = (w, z)_d$, which is a contradiction. Thus, $u_1 \neq w_1$. Since $h_l(u_1) = h_l(w_1)$, we have $|u_1 - w_1| \geqq 2l + 1$. Assume without loss of generality that $u_1 \geqq w_1 + 2l + 1$. Since $dist(\nu(u), \nu(v)) \leqq l$, we have $|u_1 - v_1| \leqq l$, and so $\min\{u_1\beta + f_u^+(v),\ v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1\} \geqq (u_1 - l)\beta + \gamma + 1$. Since $dist(\nu(w), \nu(z)) \leqq l$, we have $|w_1 - z_1| \leqq l$, and so $\max\{w_1\beta + f_w^+(z),\ z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1\} \leqq (w_1 + l)\beta + 2(l+1)\gamma \leqq (u_1 - l - 1)\beta + 2(l+1)\gamma = (u_1 - l)\beta - 1$. Thus, $\min\{u_1\beta + f_u^+(v),\ v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1\} > \max\{w_1\beta + f_w^+(z),\ z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1\}$, which is contradicting to the assumption that a segment in $P(u, v)$ and a segment in $P(w, z)$ intersect. □

Claim 4: seg_2 and seg_4 are disjoint.

Proof of Claim 4: Assume that seg_2 and seg_4 intersect. Then, $v_2\beta + f_v^-(u) = z_2\beta + f_z^-(w)$, that is $(v_2 - z_2)\beta + \{f_v^-(u) - f_z^-(w)\} = 0$. Since $|f_v^-(u) - f_z^-(w)| \leqq \gamma - 1$, we have $v_2 = z_2$ and $f_v^-(u) = f_z^-(w)$ by Lemma 10. If $v_1 = z_1$ then $v = z$. Since $f_v^-(u) = f_z^-(w)$, we have $(u, v)_d = (w, z)_d$, which is a contradiction. Thus, $v_1 \neq z_1$. Assume without loss of generality that $v_1 \geqq z_1 + 1$. Since $v_1\beta + \gamma > z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1$ by Lemma 9, we have

$$\begin{aligned} &v_1\beta + h_l(v_2)\gamma + f_v^-(u) + 1 \\ &> v_1\beta + \gamma \\ &> z_1\beta + h_l(z_2)\gamma + f_z^-(w) + 1 \\ &> z_1\beta + \gamma, \end{aligned}$$

which is contradicting to the assumption that seg_2 and seg_4 intersect. □

By Claims 1–4, we conclude that if $(u, v)_d \neq (w, z)_d$ then a segment in $P(u, v)$ has no intersection with a segment in $P(w, z)$. □

Theorem 8: $G_t(m)$ has a layout with area

$$O(t^3|V(G_t(m))|).$$

Proof: $\Delta(G_t(m)) \leqq 2t + 4$ by Property 1. From Lemma 4 and Theorem 6, we have $A_{\psi_t}(G_t(m)) = O(|V(G_t(m))|)$ and $leng_{\psi_t}(G_t(m)) = O(\sqrt{t})$. Thus, by Lemma 11, $G_t(m)$ has a layout with area $O(t^3|V(G_t(m))|)$. □

5. Layout of $G_1(m)$

In this section, we show another layout of $G_1(m)$. The upper and lower bounds for $leng(G_1(m))$ shown in this section are stronger than those derived from general bounds for $leng(G_t(m))$ shown in Theorems 6 and 7.

5.1 Embedding φ

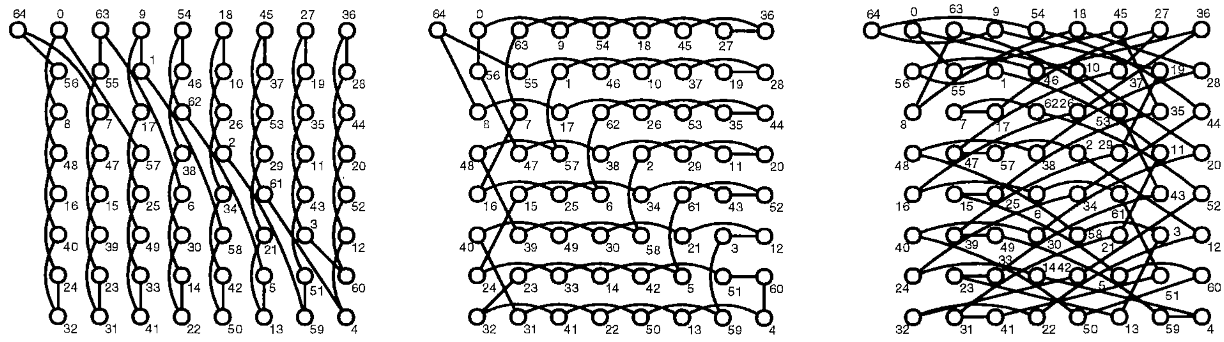
For $i \in [m^2]$, let $x(i) = i \bmod m$ and $y(i) = (\lfloor \frac{i}{m} \rfloor - i) \bmod m$. We define an embedding φ of $G_1(m)$ as follows: $\varphi(i) = [\varphi_m(x(i)) + 1, \varphi_m(y(i))]$ if $i \in [m^2]$, and $\varphi(i) = [0, 0]$ if $i = m^2$. Figure 5 shows embedding φ of $G_1(8)$.

5.2 Wire Length of φ

Theorem 9: $leng_\varphi(G_1(m)) \leqq 6$.

Proof: We will show that $dist(\varphi(i), \varphi(j)) \leqq 6$ for any $(i, j) \in E(G_1(m))$. Assume without loss of generality that $j = (i + k) \bmod (m^2 + 1)$ for $k = m, m + 1$, or $m + 2$ if $(i, j) \in E(G_1(m))$.

First, consider the case of $j = (i + m) \bmod (m^2 + 1)$. If $i = 0, \ldots, m^2 - m - 1$, then $x(j) = x(i)$ and $y(j) = (y(i) + 1) \bmod m$. Thus, $dist(\varphi(i), \varphi(j)) \leqq 2$ by Lemma 3. If $i = m^2 - m$, then $\varphi(i) = \varphi(m^2 - m) = [1, 1]$ and $\varphi(j) = \varphi(m^2) = [0, 0]$. Thus, $dist(\varphi(i), \varphi(j)) =$



(i) The edges of offset 8 (ii) The edges of offset 9 (iii) The edges of offset 10

Fig. 5 Embedding φ of $G_1(8)$.

2. If $i = m^2 - m + 1, \ldots, m^2 - 1$, then $x(j) = (x(i) - 1) \bmod m$ and $y(j) = (y(i) + 2) \bmod m$. Thus, $dist(\varphi(i), \varphi(j)) \leqq 2 + 4 = 6$ by Lemma 3. If $i = m^2$, then $\varphi(i) = \varphi(m^2) = [0, 0]$ and $\varphi(j) = \varphi(m-1) = [2, 2]$. Thus, $dist(\varphi(i), \varphi(j)) = 4$.

Next, consider the case of $j = (i+m+1) \bmod (m^2+1)$. If $i = 0, \ldots, m^2 - m - 2$, then $x(j) = (x(i) + 1) \bmod m$ and $y(j) = y(i)$ or $(y(i) + 1) \bmod m$. Thus, $dist(\varphi(i), \varphi(j)) \leqq 2 + 2 = 4$. If $i = m^2 - m - 1$, then $\varphi(i) = \varphi(m^2-m-1) = [2, 1]$ and $\varphi(j) = \varphi(m^2) = [0, 0]$. Thus, $dist(\varphi(i), \varphi(j)) = 3$. If $i = m^2 - m, \ldots, m^2 - 1$, then $x(j) = x(i)$ and $y(j) = (y(i) + 1) \bmod m$. Thus, $dist(\varphi(i), \varphi(j)) \leq 2$ by Lemma 3. If $i = m^2$, then $\varphi(i) = \varphi(m^2) = [0, 0]$ and $\varphi(j) = \varphi(m) = [1, 2]$. Thus, $dist(\varphi(i), \varphi(j)) = 3$.

Finally, consider the case of $j = (i + m + 2) \bmod (m^2 + 1)$. If $i = 0, \ldots, m^2 - m - 3$, then $x(j) = (x(i) + 2) \bmod m$ and $y(j) = (y(i) - 1) \bmod m$ or $y(i)$. Thus, $dist(\varphi(i), \varphi(j)) \leq 4 + 2 = 6$ by Lemma 3. If $i = m^2 - m - 2$, then $\varphi(i) = \varphi(m^2 - m - 2) = [4, 0]$ and $\varphi(j) = \varphi(m^2) = [0, 0]$. Thus, $dist(\varphi(i), \varphi(j)) = 4$. If $i = m^2 - m - 1$, then $\varphi(i) = \varphi(m^2 - m - 1) = [2, 1]$ and $\varphi(j) = \varphi(0) = [1, 0]$. Thus, $dist(\varphi(i), \varphi(j)) = 2$. If $i = m^2 - m, \ldots, m^2 - 2$, then $x(j) = (x(i) + 1) \bmod m$ and $y(j) = y(i)$. Thus, $dist(\varphi(i), \varphi(j)) \leqq 2$ by Lemma 3. If $i = m^2 - 1$, then $\varphi(i) = \varphi(m^2 - 1) = [2, 0]$ and $\varphi(j) = \varphi(m) = [1, 2]$. Thus, $dist(\varphi(i), \varphi(j)) = 3$. If $i = m^2$, then $\varphi(i) = \varphi(m^2) = [0, 0]$ and $\varphi(j) = \varphi(m+1) = [3, 0]$. Thus, $dist(\varphi(i), \varphi(j)) = 3$. □

5.3 Lower Bound for Wire Length

It is easy to see the following lemma.

Lemma 12: If $i = \{(m + 2)x + (m + 1)y + mz\} \bmod (m^2+1)$ for some integers x, y, and z, then the distance from 0 to i in $G_1(m)$ is at most $|x| + |y| + |z|$. □

For any graph G, let $diam(G)$ denote the diameter of G, that is the maximum distance between two vertices of G.

Lemma 13: For any m even, $diam(G_1(m)) \leqq \frac{m+2}{2}$.

Proof: Since $G_1(m)$ is vertex symmetric, it suffices to consider the distance from 0 to i for any $i \in [\frac{m^2}{2}+1]$. Let $j = \lfloor \frac{i}{m} \rfloor$, $k = \lfloor \frac{i \bmod m}{2} \rfloor$, and $l = (i \bmod m) \bmod 2 = i \bmod 2$. Then $j \leqq \frac{m}{2}$, $k \leqq \frac{m}{2} - 1$, $l \leqq 1$, and $i = jm+2k+l = (m+2)k+(m+1)l+(j-k-l)m$. Thus, by Lemma 12, we have that if $j \geqq k+l$ then the distance from 0 to i is at most $k+l+(j-k-l) = j \leqq \frac{m}{2}$, and that if $2(k+l) - \frac{m}{2} - 1 \leq j \leq k+l-1$ then the distance from 0 to i is at most $k+l+(k+l-j) = 2(k+l)-j \leqq \frac{m}{2}+1$. Since $i = -(m+2)(\frac{m}{2}-k-l)-(m+1)l+m(j+\frac{m}{2}+1-k)$, we conclude that if $j \leqq 2(k+l) - \frac{m}{2} - 2$ then the distance from 0 to i is at most $(\frac{m}{2} - k - l) + l + (j + \frac{m}{2} + 1 - k) = j + m + 1 - 2k \leqq \frac{m}{2} + 2l - 1 \leqq \frac{m}{2} + 1$ by Lemma 12. □

Lemma 14: For any m odd, $diam(G_1(m)) \leqq \frac{m+1}{2}$.

Proof: Since $G_1(m)$ is vertex symmetric, it suffices to consider the distance from 0 to i for any $i \in [\frac{m^2+1}{2}+1]$. Let $j = \lfloor \frac{i}{m} \rfloor$, $k = \lfloor \frac{i \bmod m}{2} \rfloor$, and $l = (i \bmod m) \bmod 2 = i \bmod 2$. Then $j \leqq \frac{m-1}{2}$, $k \leqq \frac{m-1}{2}$, $l \leqq 1$, and $i = jm+2k+l = (m+2)k+(m+1)l+(j-k-l)m$. Thus, by Lemma 12, we have that if $j \geqq k+l$ then the distance from 0 to i is at most $k + l + (j - k - l) = j \leqq \frac{m-1}{2}$, and that if $2(k + l) - \frac{m+1}{2} \leqq j \leqq k + l - 1$ then the distance from 0 to i is at most $k + l + (k + l - j) = 2(k + l) - j \leqq \frac{m+1}{2}$. Since $i = -(m + 2)(\frac{m+1}{2} - k - l) + (m+1)(1 - l) + m(j + \frac{m+1}{2} - k)$, we conclude that if $j \leqq 2(k + l) - \frac{m+1}{2} - 1$ then the distance from 0 to i is at most $(\frac{m+1}{2} - k - l) + (1 - l) + (j + \frac{m+1}{2} - k) = j + m + 2 - 2(k + l) \leqq \frac{m+1}{2}$ by Lemma 12. □

The following lemma is well-known and can be verified easily.

Lemma 15: Let $\mathcal{A} \subset \mathbb{N}^2$ and let d be a positive integer. If $dist(x, y) \leq d$ for any $x, y \in \mathcal{A}$ then $|\mathcal{A}| \leqq \frac{(d+1)^2+1}{2}$.

Proof: Let $\mathcal{A}_{max}$ be a maximal subset of $\mathbb{N}^2$ such that $dist(u, v) \leqq d$ for any $u, v \in \mathcal{A}_{max}$. Then, there exists a pair $x, y \in \mathcal{A}_{max}$ such that $dist(x, y) = d$. Assume without loss of generality that $x = [x_1, x_2]$, $y = [x_1 + c, x_2 + d - c]$ where $0 \leqq c \leqq d$. If $(z_1, z_2) \in \mathcal{A}_{max}$ then the following two inequalities hold:

$$z_1 + z_2 \leqq x_1 + x_2 + d, \quad (1)$$
$$z_1 + z_2 \geqq x_1 + x_2. \quad (2)$$

Let $a = [a_1, a_2]$ be a lattice point in $\mathcal{A}_{max}$ such that

$a_2 - a_1$ is minimum. Then, for any $(z_1, z_2) \in \mathcal{A}_{max}$,

$$z_2 - z_1 \geqq a_2 - a_1. \tag{3}$$

Since $dist(a, z) \leqq d$ for any $z = [z_1, z_2] \in \mathcal{A}_{max}$,

$$z_2 - z_1 \leqq a_2 - a_1 + d. \tag{4}$$

Since the number of lattice points satisfying all inequarities (1)–(4) is at most $\frac{(d+1)^2+1}{2}$, we conclude that

$$|\mathcal{A}_{max}| \leqq \frac{(d+1)^2+1}{2}.$$

□

Theorem 10: $leng(G_1(m)) \geqq \frac{2(\sqrt{2m^2+1}-1)}{m+2}$. In particular, $leng(G_1(m)) \geqq 3$ if $m \geqq 8$.

Proof: Consider any embedding η' of $G_1(m)$. If $dist(\eta'(u), \eta'(v)) \leqq d$ for any $u, v \in V(G_1(m))$ then $|\eta'(V(G_1(m)))| = m^2 + 1 \leqq \frac{(d+1)^2+1}{2}$ by Lemma 15, and so $d \geqq \sqrt{2m^2+1} - 1$. Thus, by Lemmas 13 and 14, $leng(G_1(m)) \geqq \frac{2(\sqrt{2m^2+1}-1)}{m+2}$.

If $m \geqq 8$ then $\frac{2(\sqrt{2m^2+1}-1)}{m+2} > 2$. Since $leng(G_1(m))$ is an integer, we conclude that $leng(G_1(m)) \geqq 3$. □

5.4 Layout of $G_1(m)$

Based on the embedding φ defined in 5.1, $G_1(m)$ can be laid out quite efficiently. Figure 6 shows an efficient layout of $G_1(8)$. In general, if m is a multiple of 4, $G_1(m)$ can be laid out efficiently by composing 12 building blocks shown in Figs. 8–19 as shown in Fig. 7. The area and the maximal wire length of the resulting layout of $G_1(m)$ is $72m^2 + 18m (= O(|V(G_1(m))|))$ and 6, respectively.

Acknowledgements

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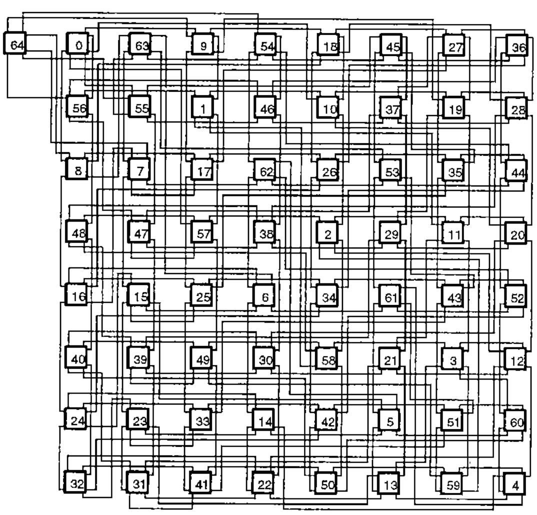

Fig. 6 A layout of $G_1(8)$.

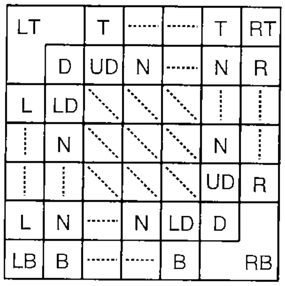

Fig. 7 A layout of $G_1(m)$.

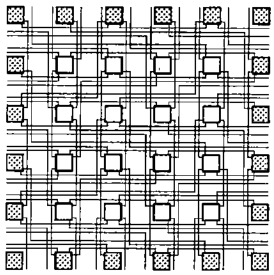

Fig. 8 Building block N.

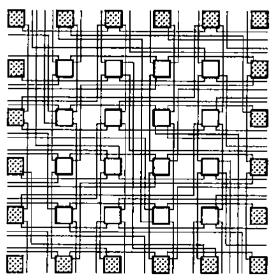

Fig. 9 Building block D.

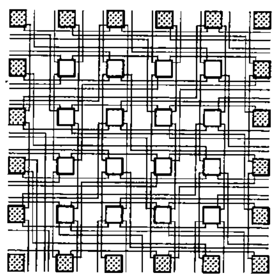

Fig. 10 Building block UD.

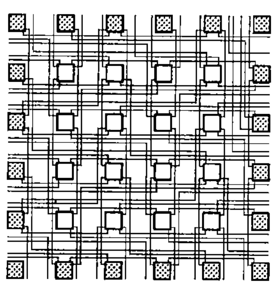

Fig. 11 Building block LD.

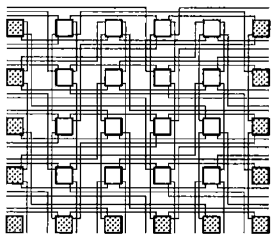

Fig. 12 Building block T.

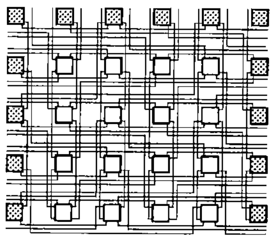

Fig. 13 Building block B.

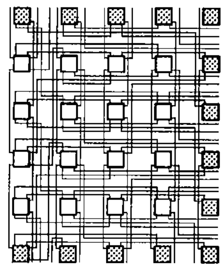

Fig. 14 Building block L.

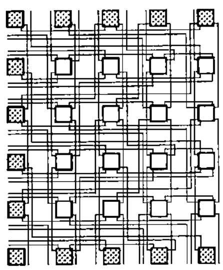

Fig. 15 Building block R.

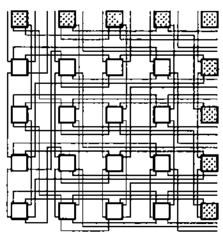

Fig. 16 Building block LB.

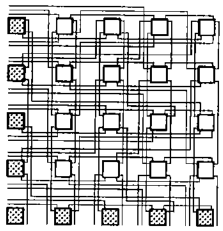

Fig. 17 Building block RT.

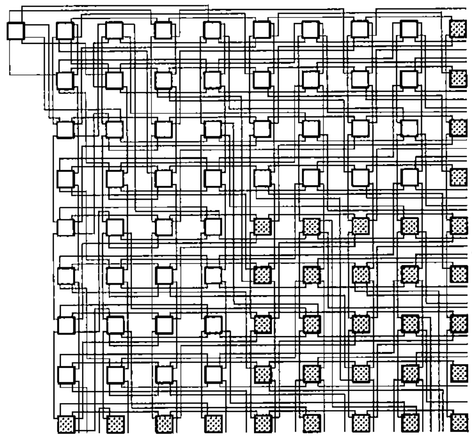

Fig. 18 Building block LT.

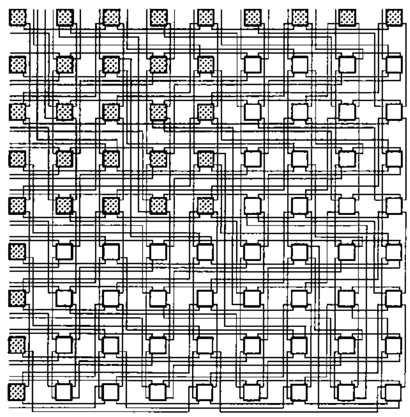

Fig. 19 Building block RB.

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