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A Linear Time Algorithm for Constructing Proper-Path-Decomposition of Width Two

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SUMMARY The problem of constructing the proper-path-decomposition of width at most 2 has an application to the efficient graph layout into ladders. In this paper, we give a linear time algorithm which, for a given graph with maximum vertex degree at most 3, determines whether the proper-pathwidth of the graph is at most 2, and if so, constructs a proper-path-decomposition of width at most 2.

key words: *proper-path-decomposition, proper-pathwidth, pathwidth, graph layout*

1. Introduction

The pathwidth of a graph G is the minimum value of k such that G can be obtained from a sequence of graphs $H_1, H_2, \ldots, H_r$ each of which has at most $k+1$ vertices, by identifying some vertices of H_i pairwise with some of $H_{i+1} (1 \leqq i < r)$ [5]. The sequence $H_1, H_2, \ldots, H_r$ is called a path-decomposition of G with width k. The proper-pathwidth is introduced in [6] as a variant of the pathwidth. The (proper-)pathwidth is closely related to other graph parameters such as cutwidth, topological bandwidth, and search numbers. It is NP-complete to decide, given a graph G and an integer k, whether the (proper-)pathwidth of G is at most k, while the problem is in P if k is a fixed integer. It is shown in [2] that if the pathwidth of a graph G is bounded by a fixed integer k then a path-decomposition of G with width k can be constructed in polynomial time. On the other hand, no polynomial time algorithm is known for the problem of constructing a proper-path-decomposition of width k for a graph with proper-pathwidth bounded by a fixed integer $k \geqq 2$.

The graphs which can be laid out into ladders are characterized in terms of the proper-pathwidth of graphs [3]. It is known that finding a proper-path-decomposition of width 2 for a graph with maximum vertex degree 3 and proper-pathwidth 2 is crucial to lay out such a graph into the ladder [3].

The purpose of this paper is to give a linear time algorithm for constructing a proper-path-decomposition of width 2 for a graph with maximum vertex degree 3 and proper-pathwidth 2.

It is shown in [1] that if the treewidth of a graph G is bounded by a fixed integer k then a tree-decomposition of G with width k can be constructed in linear time and, by using this fact and the result of [2], a path-decomposition of G with minimum width can also be constructed in linear time. However, this result cannot be generalized immediately to our problem of constructing proper-path-decompositions of minimum width since there exist graphs with the proper-pathwidth more than the pathwidth because of an additional condition ((e) in Condition 1 given in Sect. 2) which is introduced to define the proper-path-decomposition.

The rest of the paper is organized as follows. Some definitions are given in Sect. 2. In Sect. 3, we give a characterization of graphs with maximum vertex degree 3 and proper-pathwidth 2. We give in Sect. 4 the proof of the characterization and an algorithm for constructing a proper-path-decomposition of width 2.

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2. Preliminaries

Let G be a graph and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. $\Gamma_G(v)$ is the set of edges incident to a vertex v in G. $|\Gamma_G(v)|$ is called the *degree* of v and denoted by $\deg_G(v)$. Let $\Delta(G) = \max\{\deg_G(v) \mid v \in V(G)\}$. $N_G(v)$ is the set of vertices adjacent to a vertex v in G. For $U \subseteqq V(G)$, let $G[U]$ be the subgraph of G induced by U, and let $G-U$ denote $G[V(G)-U]$. Similarly, for $S \subseteqq E(G)$, let $G[S]$ be the subgraph of G induced by S, and let $G-S$ denote the graph obtained from G by deleting S. For graphs G and H, $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and $G \cap H$ is the graph with vertex set $V(G) \cap V(H)$ and edge set $E(G) \cap E(H)$. Although a path is a graph, we often denote a path by a sequence of vertices in which consecutive two vertices are adjacent in the path.

A vertex v of G is a *cut vertex* if $E(G)$ can be partitioned into two nonempty subsets E_1 and E_2 such that $G[E_1]$ and $G[E_2]$ have just the vertex v in common. A connected graph that has no cut vertices is called a *block*. Every block with at least three vertices is 2-*connected*. A *block of a graph* is a subgraph that is a block and is maximal with respect to this property.

A graph is *outer planar* if it has a planar drawing in which the outer region includes all of its vertices. An edge is *outer* if it is included in the outer region, and is *inner* otherwise. A cycle C of an outer planar graph

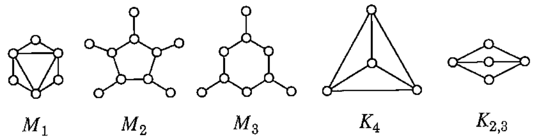

Fig. 1 Minimal forbidden minors for $\mathcal{P}_2$.

G is an *end-region* of G if $C = G[V(C)]$ and C has at most one inner edge. Any 2-connected outer planar graph has at least one end-region, and it has at least two end-regions if it has an inner edge.

For a graph G, a sequence $\mathcal{X} = (X_1, \ldots, X_r)$ of subsets of $V(G)$ is called a *proper-path-decomposition* of G if $\mathcal{X}$ satisfies the following conditions.

Condition 1:

(a) $X_i \not\subseteq X_j$ $(i \neq j)$;

(b) $\bigcup_{1 \leqq i \leqq r} X_i = V(G)$;

(c) for any $(u, v) \in E(G)$, there exists an i such that $u, v \in X_i$;

(d) for all a, b, and c with $1 \leqq a \leqq b \leqq c \leqq r$, $X_a \cap X_c \subseteq X_b$;

(e) for all a, b, and c with $1 \leqq a < b < c \leqq r$, $|X_a \cap X_c| \leqq |X_b| - 2$ if $|X_b| \geqq 2$.

The *width* of $\mathcal{X}$ is $\max_{1 \leqq i \leqq r} |X_i| - 1$. The *proper-pathwidth* of G is the minimum width over all proper-path-decompositions of G, and denoted by $ppw(G)$. A proper-path-decomposition is said to be *optimal* if it has width of $ppw(G)$. A proper-path-decomposition of width k is called a k-proper-path-decomposition.

A graph H is a *minor* of a graph G if H is isomorphic to a graph obtained from a subgraph of G by contracting edges. A family $\mathcal{F}$ of graphs is said to be *minor-closed* if the following condition holds: If $G \in \mathcal{F}$ and H is a minor of G then $H \in \mathcal{F}$. A graph G is a *minimal forbidden minor* for a minor-closed family $\mathcal{F}$ of graphs if $G \notin \mathcal{F}$ and any proper minor of G is in $\mathcal{F}$. $\mathcal{F}$ is characterized by the minimal forbidden minors for $\mathcal{F}$. That is, a graph G is in $\mathcal{F}$ if and only if no minimal forbidden minor for $\mathcal{F}$ is a minor of G. For a positive integer k, the family $\mathcal{P}_k$ of graphs with proper-pathwidth at most k is minor-closed. K_3 and $K_{1,3}$ are the minimal forbidden minors for $\mathcal{P}_1$ [6], and 36 graphs are known as the minimal forbidden minors for $\mathcal{P}_2$ [7]. The five minimal forbidden minors for $\mathcal{P}_2$ shown in Fig. 1 will be used in Sect. 4.

3. Characterization

In this section, we characterize graphs with maximum vertex degree 3 and proper-pathwidth 2.

Suppose that G' is a graph obtained from a graph G by deleting self-loops and replacing multiple edges with a single edge. A proper-path-decomposition of G' is also that of G, and vice versa, by definition. Therefore, an optimal proper-path-decomposition of G' is also that of G. An optimal proper-path-decomposition of a graph can be obtained by concatenating optimal proper-path-decompositions of connected components. From these facts, we assume that the graphs considered in the rest of the paper are simple and connected.

A cut vertex of a graph G is called a *connection point* of G if the vertex is contained in a 2-connected block of G. Since a connection point of G is a cut vertex of G, $E(G)$ can be partitioned into disjoint sets $E_1, \ldots, E_l$ such that $G[E_i]$ and $G[E_j]$ share at most one connection point of G for any i and j with $1 \leqq i < j \leqq l$. Let $\mathcal{D} = \{G[E_i] \mid 1 \leqq i \leqq l\}$. We define that $\mathcal{H}$ is the set of 2-connected components in $\mathcal{D}$. A component of $\mathcal{D} - \mathcal{H}$ is called a *path component* of G if the component is a path. $\mathcal{P}$ denotes the set of path components of G. A component of $\mathcal{D} - (\mathcal{H} \cup \mathcal{P})$ is called a *tree component* of G. $\mathcal{T}$ denotes the set of tree components of G.

The following characterization for trees with proper-pathwidth at most k is given in [9].

Lemma A: For a tree T and an integer $k \geqq 2$, $ppw(T) \leqq k$ if and only if there exists a path P in T such that $ppw(T - V(P)) \leqq k - 1$. □

k-spine of T is a path satisfying the condition of Lemma A.

The following is the main theorem of the paper.

Theorem 1: For a graph G with $\Delta(G) \leqq 3$, $ppw(G) \leqq 2$ if and only if G has a sequence $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of distinct components in $\mathcal{D}$ and a sequence $\mathcal{A} = (a_0, a_1, \ldots, a_m)$ of distinct vertices of G such that the following condition is satisfied. Let $\mathcal{D}' = \mathcal{D} - \{C_i \mid 1 \leqq i \leqq m\}$.

Condition 2:

(a) $V(C_i) \cap V(C_{i+1}) = \{a_i\}$ for $1 \leqq i < m$, $a_0 \in V(C_1)$, and $a_m \in V(C_m)$.

(b) $\deg_G(a_0) \leqq 2$ and $\deg_G(a_m) \leqq 2$.

(c) For $1 \leqq i \leqq m$, if $C_i \in \mathcal{T}$ then the path in C_i connecting a_{i-1} and a_i is a 2-spine of C_i.

(d) For $1 \leqq i \leqq m$, if $C_i \in \mathcal{H}$ then C_i is an outer planar graph with at most two end-regions. Moreover, each end-region contains a_{i-1} or a_i.

(e) $\mathcal{D}' \subseteq \mathcal{P}$.

(f) There exists a one-to-one mapping $f : \mathcal{D}' \rightarrow \{i \mid 1 \leqq i \leqq m\} \times \{0, 1\}$ satisfying the following statement.

For $P \in \mathcal{D}'$, $f(P) = (i, j)$ if and only if $C_i \in \mathcal{H}$ and there exists an end-vertex x of P such that $(x, a_{i-j}) \in E(C_i)$. $(*)$

□

In the following section, we give a constructive proof for Theorem 1, and based on the proof, we describe a linear time algorithm which, given a graph G with $\Delta(G) \leqq 3$, determines whether $ppw(G) \leqq 2$, and if so, constructs a proper-path-decomposition of width at most 2 of G.

4. Proof and Algorithm

We first prove the theorem for a special case of $|\mathcal{D}| = 1$. We prove the theorem for trees and 2-connected graphs in Sect. 4.1 and 4.2, respectively. The proof for general case is given in Sect. 4.3. We also give in Sect. 4.3 an algorithm for general graphs.

For a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of elements, X_1 and X_r are called the *head* of $\mathcal{X}$ and its *tail*, respectively. We denote the sequence without elements by *nul*. For sequences $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ and $\mathcal{Y} = (Y_1, Y_2, \ldots, Y_q)$, we define that $\mathcal{X} + \mathcal{Y} = (X_1, X_2, \ldots, X_r, Y_1, Y_2, \ldots, Y_q)$. For a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_r)$ of subsets of a set Ω and $W \subseteq \Omega$, we define that $\mathcal{X} \cup W = (X_1 \cup W, X_2 \cup W, \ldots, X_r \cup W)$ and $\mathcal{X} \cap W = (X_1 \cap W, X_2 \cap W, \ldots, X_r \cap W)$.

4.1 Binary Trees

Theorem 1 is immediate for binary trees by Lemma A. An algorithm for constructing optimal proper-path-decompositions of trees is shown in [8]. Since this algorithm computes $ppw(T)$ in $O(N)$ time for an N-vertex tree T and provides an optimal proper-path-decomposition of T in $O(N ppw(T))$ time, we can construct a 2-proper-path-decomposition of T with $ppw(T) = 2$ in linear time.

In this subsection, we show algorithms for constructing a proper-path-decomposition of a binary tree with width at most 2 satisfying some conditions. These algorithms will be used to construct an algorithm for general graphs.

Lemma 2: For a path $P = (p_0, \ldots, p_l)$, there exists a 1-proper-path-decomposition $\mathcal{X} = (X_1, \ldots, X_r)$ of P such that $p_0 \in X_1$ and $p_l \in X_r$.

Proof: Let $\mathcal{X} = (X_1, \ldots, X_l)$ with $X_i = \{p_{i-1}, p_i\}$ $(1 \leqq i \leqq l)$ if $l \geqq 1$, $\mathcal{X} = (\{p_0\})$ otherwise. $\mathcal{X}$ is clearly a desired proper-path-decomposition. □

Algorithm PPD_PATH shown in Fig. 2 is the formal description of the procedure written in the proof of Lemma 2. Trivially, PPD_PATH can be executed in linear time.

```
Procedure PPD_PATH ( P )
  [ Input:  a path P = (p_0, p_1, ..., p_l);
    Output: a 1-proper-path-decomposition (X_1, X_2, ..., X_r)
            of P such that p_0 ∈ X_1 and p_l ∈ X_r; ]
  1. if l = 0 then return ({p_0});
  2. for each 1 ≦ i ≦ l do
          X_i := {p_{i-1}, p_i};
     endfor ;
  3. return (X_1, X_2, ..., X_l);
End
```

Fig. 2 Algorithm for constructing a 1-proper-path-decomposition of a path.

Lemma 3: For a binary tree T with $ppw(T) = 2$ and its 2-spine $P = (p_0, \ldots, p_l)$ such that $\deg_T(p_0) = \deg_T(p_l) = 1$, there exists a 2-proper-path-decomposition $\mathcal{X} = (X_1, \ldots, X_r)$ of T such that $p_0 \in X_1 - \bigcup_{1<i\leqq r} X_i$ and $p_l \in X_r - \bigcup_{1\leqq i<r} X_i$.

Proof: Since P is a 2-spine of T, it follows from Lemma A that $ppw(T - V(P)) \leqq 1$. Thus, each connected component of $T - V(P)$ is a path. For $0 < i < l$, at most one connected component P_i of $T - V(P)$ has a vertex adjacent to p_i since $\Delta(T) \leqq 3$. Let $I = \{i \mid 0 < i < l, \deg_T(p_i) = 3\}$. We define the sequence $\mathcal{X}$ of subsets of $V(T)$ as follows:

$$\mathcal{X} = (S_1) + \mathcal{Y}_1 + (S_2) + \cdots + (S_{l-1}) + \mathcal{Y}_{l-1} + (S_l),$$

where for $1 \leqq i \leqq l$,

$$S_i = \begin{cases} \{p_{i-1}, p_i\} \cup V(P_i) & \text{if } i \in I \text{ and } |V(P_i)| = 1 \\ \{p_{i-1}, p_i\} & \text{otherwise} \end{cases}$$

for $1 \leqq i < l$,

$$\mathcal{Y}_i = \begin{cases} \text{PPD\_PATH}(P_i) \cup \{p_i\} & \text{if } i \in I \text{ and } |V(P_i)| \geqq 2 \\ nul & \text{otherwise} \end{cases}$$

We show that $\mathcal{X}$ is a desired 2-proper-path-decomposition. The following claim can be easily observed from the definition of $\mathcal{X}$.

Claim 4:

1. p_0 and p_l appear in S_1 and S_l, respectively.
2. For $0 < i < l$, p_i appears in $S_i \cap S_{i+1}$. Moreover, p_i appears in every element of $\mathcal{Y}_i$ if $\mathcal{Y}_i \neq nul$.
3. For $i \in I$ with $|V(P_i)| \geqq 2$, $v \in V(P_i)$ appears in at most two consecutive elements of $\mathcal{Y}_i$.
4. For $i \in I$ with $|V(P_i)| = 1$, $v \in V(P_i)$ appears in S_i.

It is clear by Claim 4 that $\mathcal{X}$ satisfies (a), (b), and (c) in Condition 1. Moreover, $\mathcal{X}$ satisfies (d) in Condition 1 since we can observe that any vertex of T appears in consecutive elements of $\mathcal{X}$. In what follows, we show that $\mathcal{X}$ satisfies (e) in Condition 1. If $X_a \cap X_c = \emptyset$ for all a and c with $1 < a+1 \leqq c-1 < r$ then the condition is clearly satisfied. Thus, we assume that there exist a and c with $1 < a+1 \leqq c-1 < r$ such that $X_a \cap X_c \neq \emptyset$. Since any vertex in $V(T) - \{p_i \mid i \in I, |V(P_i)| \geqq 2\}$ appears in at most two consecutive elements of $\mathcal{X}$, there exists p_i such that $i \in I$, $|V(P_i)| \geqq 2$, and $p_i \in X_a \cap X_c$. Since $(X_a, \ldots, X_c)$ is a subsequence of $(S_i) + \mathcal{Y}_i + (S_{i+1})$,

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Procedure PPD_TREE ( T, P )
  [ Input:  a binary tree T;
            a 2-spine P = (p_0, ..., p_l) of T such that
            deg_T(p_0) = deg_T(p_l) = 1;
    Output: a proper-path-decomposition (X_1, ..., X_r) of T
            with width at most 2 such that p_0 ∈ X_1 − ⋃_{1<i≦r} X_i
            and p_l ∈ X_r − ⋃_{1≦i<r} X_i; ]
  1. for i := 1 to l − 1 do
       a. S_i := {p_{i−1}, p_i};
       b. Y_i := nul;
       c. if deg_T(p_i) = 3 then
            i. let P_i be the connected component in T − V(P)
               which has a vertex adjacent to p_i in T;
           ii. if |V(P_i)| = 1 then S_i := {p_{i−1}, p_i} ∪ V(P_i);
               else Y_i := PPD_PATH(P_i) ∪ {p_i};
     endfor ;
  2. S_l := {p_{l−1}, p_l};
  3. return (S_1) + Y_1 + (S_2) + ··· + (S_{l−1}) + Y_{l−1} + (S_l);
End
```

Fig. 3 Algorithm for constructing a 2-proper-path-decomposition of a binary tree with its 2-spine.

no vertices in $V(P) - \{p_i\}$ are contained in $X_a \cap X_c$. Moreover, since X_b is an element of $\mathcal{Y}_i$ for any b with $a < b < c$, it follows from $|V(P_i)| \geqq 2$ that $|X_b| = 3$. Thus, we have that $|X_a \cap X_c| = |\{p_i\}| = 1 \leqq |X_b| - 2$ for any b with $a < b < c$. Therefore, $\mathcal{X}$ satisfies (e) in Condition 1. It is clear that the width of $\mathcal{X}$ is at most 2 and that $p_0 \in X_1 - \bigcup_{1<i\leqq r} X_i$ and $p_l \in X_r - \bigcup_{1\leqq i<r} X_i$. Therefore, $\mathcal{X}$ is a desired proper-path-decomposition. □

We describe Algorithm PPD_TREE based on Lemma 3 in Fig. 3. The following corollary is immediate.

Corollary 5: Given a binary tree T and a 2-spine $P = (p_0, \ldots, p_l)$ of T such that $\deg_T(p_0) = \deg_T(p_l) = 1$, PPD_TREE outputs in linear time a proper-path-decomposition $(X_1, \ldots, X_r)$ of T with width at most 2 such that $p_0 \in X_1 - \bigcup_{1<i\leqq r} X_i$ and $p_l \in X_r - \bigcup_{1\leqq i<r} X_i$. □

4.2 2-Connected Graphs

In this subsection, we show a necessary and sufficient condition for a 2-connected graph G to have $ppw(G) = 2$, and based on this condition, we give an algorithm for constructing a 2-proper-path-decomposition of G. This algorithm is used in the next subsection to construct an algorithm for general graphs.

Theorem 1 is immediate for 2-connected graphs by the following lemma.

Lemma 6: For a 2-connected graph G, $ppw(G) = 2$ if and only if G is outer planar and has at most two end-regions.

Proof: First, we assume that $ppw(G) = 2$. Then none of M_1, K_4, and $K_{2,3}$ which are shown in Fig. 1 is a minor of G. It is well-known that the family of outer planar graphs is minor-closed and that K_4 and $K_{2,3}$ are the minimal forbidden minors for the family of outer planar graphs. Thus G is outer planar. Moreover, G has at most two end-regions since M_1 is not a minor of G.

Next, we assume that G is outer planar and has at most two end-regions. Let e_s and e_t be any edges in G satisfying the following condition:

Condition 3: e_s and e_t are outer edges contained in distinct end-regions if G has two end-regions.

It suffices to show the following claim.

Claim 7: There exists a 2-proper-path-decomposition $\mathcal{X} = (X_1, \ldots, X_r)$ of G such that

$$|X_i| = 3 \ (1 \leqq i \leqq r), \tag{1}$$

$$e_s \in E(G[X_1]) - E(G[\bigcup_{1<i\leqq r} X_i]), \text{ and} \tag{2}$$

$$e_t \in E(G[X_r]) - E(G[\bigcup_{1\leqq i<r} X_i]). \tag{3}$$

We prove this claim by induction on $|V(G)|$.

If $|V(G)| = 3$ then $\mathcal{X} = (V(G))$ is clearly a desired proper-path-decomposition.

We assume that the claim holds for any G' with $|V(G)| - 1 \geqq 3$ vertices and for any pair of edges in G' satisfying Condition 3. Since $|V(G)| \geqq 4$, there exists a degree 2 vertex s incident to e_s but not to e_t. Suppose that $e_s = (s, y)$ and $N_G(s) - \{y\} = \{x\}$. Let G' be the graph obtained by contracting the edge (s, x). Since s is identified with x, we denote the resulting vertex of G' by x. G' is clearly an outer planar graph with at most two end-regions. By the definitions of s, x, and y, (x, y) and e_t are distinct edges in G' satisfying Condition 3. Therefore, by induction hypothesis, there exists a 2-proper-path-decomposition $\mathcal{Y} = (Y_1, \ldots, Y_l)$ of G' such that

$$|Y_i| = 3 \ (1 \leqq i \leqq l), \tag{4}$$

$$(x, y) \in E(G'[Y_1]) - E(G'[\bigcup_{1<i\leqq l} Y_i]), \text{ and} \tag{5}$$

$$e_t \in E(G'[Y_l]) - E(G'[\bigcup_{1\leqq i<l} Y_i]). \tag{6}$$

We show that $\mathcal{X} = (\{s, x, y\}) + \mathcal{Y}$ is a desired 2-proper-path-decomposition of G.

We first show that $\mathcal{X}$ satisfies (1), (2), and (3). It follows from (4) and the definition of $\mathcal{X}$ that $\mathcal{X}$ satisfies (1). Since

$$s \notin Y_i \ (1 \leqq i \leqq l), \tag{7}$$

we have that

$$e_s \in E(G[\{s, x, y\}]) - E(G[\bigcup_{1\leqq i\leqq l} Y_i]). \tag{8}$$

It follows from (6) and (8) that $\mathcal{X}$ satisfies (2) and (3).

We next show that $\mathcal{X}$ is a 2-proper-path-decomposition of G. $\mathcal{X}$ clearly satisfies (a), (b), and (c) in Condition 1. Since $\mathcal{Y}$ is a proper-path-decomposition of G' and $|Y_i| = 3$ for all i with $1 \leqq i \leqq l$, it follows that

$$Y_a \cap Y_c \subseteqq Y_b \ (1 \leqq a \leqq b \leqq c \leqq l), \qquad (9)$$
$$|Y_a \cap Y_c| \leqq |Y_b| - 2 \ (1 \leqq a < b < c \leqq l).$$

Thus, to show that $\mathcal{X}$ satisfies (d) and (e) in Condition 1, it suffices to prove that $\{s,x,y\} \cap Y_c \subseteqq Y_b$ and $|\{s,x,y\} \cap Y_c| \leqq |Y_b| - 2$ for $1 \leqq b < c \leqq l$. It follows from (5) that

$$\{x,y\} \subseteqq Y_1, \qquad (10)$$
$$\{x,y\} \nsubseteq \bigcup_{1<i\leqq l} Y_i. \qquad (11)$$

It follows from (7), (9), and (10) that $\{s,x,y\} \cap Y_c = \{x,y\} \cap Y_c \subseteqq Y_1 \cap Y_c \subseteqq Y_b$ for $1 \leqq b < c \leqq l$. It follows from (7) and (11) that $|\{s,x,y\} \cap Y_c| \leqq 1$ for $1 < c \leqq l$. Thus we have that $|\{s,x,y\} \cap Y_c| \leqq |Y_b| - 2$ for $1 \leqq b < c \leqq l$ by (4).

Therefore, $\mathcal{X}$ is a desired 2-proper-path-decomposition of G, and we conclude that the lemma holds. □

We describe in Fig. 4 Algorithm PPD_2CG based on Lemma 6.

Corollary 8: Given a 2-connected outer planar graph G with at most two end-regions and any edges e_s and e_t in G satisfying Condition 3, PPD_2CG outputs in linear time a 2-proper-path-decomposition $(X_1, \ldots, X_r)$ of G satisfying (1), (2), and (3).

Proof: The correctness of PPD_2CG is immediate from the proof of Lemma 6. PPD_2CG involves $|V(G)|$ recursive calls each of which consists of constant time operations. Therefore, PPD_2CG can be executed in linear time. □

```
Procedure PPD_2CG ( G, e_s, e_t )
  Input:  a 2-connected outer planar graph G with at most
          two end-regions;
          edges e_s and e_t satisfying Condition 3;
  Output: a 2-proper-path-decomposition (X_1,...,X_r) of
          G satisfying (1), (2), and (3);
  1. if |V(G)| = 3 then return (V(G));
  2. let s be a vertex such that deg_G(s) = 2, e_s ∈ Γ(s), and
     e_t ∉ Γ(s);
  3. let {x,y} := N_G(s) such that (s,y) = e_s;
  4. let G' be the graph obtained from G by contracting (s,x);
  5. return ({s,x,y}) + PPD_2CG(G', (x,y), e_t);
End
```

Fig. 4 Algorithm for constructing a 2-proper-path-decomposition of a 2-connected graph.

4.3 General Graphs

In this section, we prove Theorem 1 and describe our algorithm for general graphs. The following lemma will be used extensively throughout this subsection.

Lemma 9: Let $\mathcal{X} = (X_1, \ldots, X_r)$ be a 2-proper-path-decomposition of a graph G with $ppw(G) = 2$. For a path P connecting a vertex $s \in X_1$ and a vertex $t \in X_r$, every connected component of $G - V(P)$ is a path.

Proof: Suppose that $\mathcal{Y} = (Y_1, \ldots, Y_r)$ is $\mathcal{X} \cap (V(G) - V(P))$. It suffices to show that the sequence $\mathcal{Y}'$ obtained from $\mathcal{Y}$ by deleting redundant elements is a 1-proper-path-decomposition of $G - V(P)$. $\mathcal{Y}$ clearly satisfies (b), (c), and (d) in Condition 1 for $G - V(P)$. Thus, $\mathcal{Y}'$ satisfies (a), (b), (c), and (d) in Condition 1 for $G - V(P)$. To show that $\mathcal{Y}'$ satisfies (e) in Condition 1, it suffices to prove that both of the following statements holds: (i) $|Y_i| \leqq 2$ for any $1 \leqq i \leqq r$; (ii) $Y_a = Y_c$ or $|Y_a \cap Y_c| = 0$ for all a and c with $1 < a+1 \leqq c-1 < r$. Every X_i $(1 \leqq i \leqq r)$ contains a vertex of P since end-vertices s and t of P are contained in X_1 and X_t, respectively, and $\mathcal{X}$ satisfies (c) and (d) in Condition 1. Since the width of $\mathcal{X}$ is 2, we have that $|Y_i| \leqq 2$, i.e. (i) holds.

Since $\mathcal{X}$ satisfies (e) in Condition 1, we have that

$$|X_a \cap X_c| \leqq |X_b| - 2 \leqq 3 - 2 = 1 \qquad (12)$$

for any a, b, and c with $1 \leqq a < b < c \leqq r$. For a, b, and c with $1 \leqq a < b < c \leqq r$, let $p_a \in X_a \cap V(P)$, $p_b \in X_b \cap V(P)$, and $p_c \in X_c \cap V(P)$.

Case 1 $p_a = p_c$. It follows from (12) that $|X_a \cap X_c| = 1$. Thus, we have $|Y_a \cap Y_c| = 0$.

Case 2 $p_a \neq p_c$. It suffices to show that, if $|Y_a \cap Y_c| = 1$ then $Y_a = Y_c$. We assume that $|Y_a \cap Y_c| = 1$, and show that $Y_a = Y_c$. Let $v \in Y_a \cap Y_c$. It follows from (d) in Condition 1 that $v \in Y_b \subset X_b$. Now we show that $X_b - (V(P) \cup \{v\}) = \emptyset$. We prove this by contradiction. Assume that $X_b - (V(P) \cup \{v\}) \neq \emptyset$. Since $|X_b| \leqq 3$, it follows from assumption that $X_b \cap V(P) = \{p_b\}$. Since P connects $s \in X_1$ and $t \in X_r$, it follows from $1 < b < r$ that $p_b \in X_{b-1} \cap X_{b+1}$. Moreover, since $v \in Y_a \cap Y_c$ and $\mathcal{X}$ satisfies (d) in Condition 1, we have that $v \in X_{b-1} \cap X_{b+1}$. Thus, we have that $|X_{b-1} \cap X_{b+1}| \geqq |\{p_b, v\}| = 2$, contradicting (12). Therefore, it follows that $X_b - (V(P) \cup \{v\}) = \emptyset$. Since this holds for any b with $a < b < c$, we have $Y_a = Y_{a+1} = \cdots = Y_c = \{v\}$.

Therefore, (ii) holds. □

In what follows, G is a graph with $\Delta(G) = 3$. Let $\mathcal{H}, \mathcal{T}$, and $\mathcal{P}$ be the sets of 2-connected components, tree components, and path components of G, respectively, and $\mathcal{D} = \mathcal{H} \cup \mathcal{T} \cup \mathcal{P}$.

Proof of Necessity for Theorem 1

We first show the necessity. Assume that $ppw(G) = 2$. Since the theorem is proved for the case of $|\mathcal{D}| = 1$ in Sect. 4.1 and 4.2, we assume that $|\mathcal{D}| \geqq 2$. It follows from assumption that $|V(G)| \geqq 4$. There exists a 2-proper-path-decomposition $\mathcal{X} = (X_1, \ldots, X_r)$ of G. Since $\mathcal{X}$ satisfies (a) in Condition 1 and $|V(G)| \geqq 4$, there exist $s \in X_1 - X_2$ and $t \in X_r - X_{r-1}$. We define that S is a path connecting s and t.

Claim 10: For $D \in \mathcal{D}$, $D \cap S$ is connected if $D \cap S$ has a vertex.

Proof: By the definitions of 2-connected components, tree components, and path components, every path in G connecting vertices of D is a subgraph of D. Thus, the claim holds. □

Let $C_1, C_2, \ldots, C_m$ be components in $\mathcal{D}$ containing an edge of S. By Claim 10, $C_i \cap S$ is a subpath of S $(1 \leqq i \leqq m)$. Moreover, $C_i \cap S$ and $C_j \cap S$ are internally vertex-disjoint since C_i and C_j share at most one connection point for $1 \leqq i < j \leqq m$. Thus we may assume without loss of generality that $C_i \cap S$ and $C_{i+1} \cap S$ share a connection point a_i for $1 \leqq i < m$. Let $a_0 = s$ and $a_m = t$. Notice that a_{i-1} and a_i are end-vertices of $C_i \cap S$ for $1 \leqq i \leqq m$. Moreover, a_{i-1} and a_i are distinct vertices since $C_i \cap S$ has at least two vertices for $1 \leqq i \leqq m$. This means that $a_0, a_1, \ldots, a_m$ are distinct vertices of G. We define that $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ and $\mathcal{A} = (a_0, a_1, \ldots, a_m)$. We show that $\mathcal{C}$ and $\mathcal{A}$ satisfies Condition 2.

$\mathcal{C}$ and $\mathcal{A}$ clearly satisfies (a) in Condition 2 by definition. The following claim shows that $\mathcal{C}$ and $\mathcal{A}$ satisfies (b) in Condition 2.

Claim 11: $\deg_G(s) \leqq 2$ and $\deg_G(t) \leqq 2$.

Proof: $|X_1| \leqq 3$ and $|X_r| \leqq 3$ since the width of $\mathcal{X}$ is 2. Thus, we have $\deg_G(s) \leqq 2$ and $\deg_G(t) \leqq 2$ since s is only in X_1 and t is only in X_r. □

The following claim shows that $\mathcal{C}$ and $\mathcal{A}$ satisfy (c) in Condition 2.

Claim 12: If $C_i \in \mathcal{T}$ $(1 \leqq i \leqq m)$, then the path in C_i connecting a_{i-1} and a_i is a 2-spine of C_i.

Proof: Let S' be the path in C_i connecting a_{i-1} and a_i. By Lemma 9, every connected component of $G - V(S)$ is a path. Since S' is a subpath of S, every connected component of $C_i - V(S')$ is a path. This means that S' is a 2-spine of C_i. □

The following claim shows that $\mathcal{C}$ and $\mathcal{A}$ satisfy (d) in Condition 2. Let $P_i^s = (s, \ldots, a_i)$ and $P_i^t = (a_i, \ldots, t)$ be the subpaths of S for $0 \leqq i \leqq m$.

Claim 13: If $C_i \in \mathcal{H}$ $(1 \leqq i \leqq m)$, then C_i is an outer planar graph with at most two end-regions. Moreover, each end-region contains a_{i-1} or a_i.

Proof: Suppose that $C_i \in \mathcal{H}$ $(1 \leqq i \leqq m)$. Since $ppw(G) = 2$, we have that $ppw(C_i) = 2$. Thus, C_i is an outer planar graph with at most two end-regions from Lemma 6. It remains to show that each end-region of C_i contains a_{i-1} or a_i. If C_i has an end-region Z which contains neither a_{i-1} nor a_i, then there exists a path $\overline{P}$ in C_i which connects a_{i-1} and a_i and contains no vertices in Z. $S' = P_{i-1}^s \cup \overline{P} \cup P_i^t$ is clearly a path connecting s and t. Since S' and Z are vertex-disjoint, $G - V(S')$ contains a cycle as a subgraph. However, this contradicts Lemma 9. Thus, each end-region of C_i contains a_{i-1} or a_i. □

The following claim shows that $\mathcal{C}$ and $\mathcal{A}$ satisfy (e) in Condition 2. Let $\mathcal{D}' = \mathcal{D} - \{C_i \mid 1 \leqq i \leqq m\}$.

Claim 14: $\mathcal{D}' \subseteqq \mathcal{P}$.

Proof: We show that any $D \in \mathcal{H} \cup \mathcal{T}$ is an element of $\mathcal{C}$. By Claim 10 and the definition of $\mathcal{C}$, it suffices to show that $|V(D \cap S)| \geqq 2$. By Lemma 9, $D \cap S$ has at least one vertex. Thus it remains to show that $|V(D \cap S)| \neq 1$. We prove this by contradiction. Assume that $V(D \cap S) = \{x\}$.

Case 1 $D \in \mathcal{H}$. If $x \in V(S) - \{s, t\}$ then we have $\deg_G(x) = \deg_D(x) + \deg_S(x) \geqq 2 + 2 = 4$, which is a contradiction since $\Delta(G) = 3$. If $x \in \{s, t\}$ then we have $\deg_G(x) = \deg_D(x) + \deg_S(x) \geqq 2 + 1 = 3$, which also contradicts Claim 11.

Case 2 $D \in \mathcal{T}$. Since there exists an edge in $\Gamma_S(x)$ which is not contained in D, x is a connection point of G. Thus, there exists $H \in \mathcal{H}$ containing x. Since H is 2-connected and $\Delta(G) = 3$, x is incident to just two edges of H and to exactly one edge of D. Thus, it follows from Claim 11 that $x \notin \{s, t\}$ and S has two edges in $\Gamma_H(x)$. This means that H is an element of $\mathcal{C}$ and $x \notin \{a_i \mid 0 \leqq i \leqq m\}$. Suppose that $H = C_i$ $(1 \leqq i \leqq m)$. Since C_i is 2-connected, there exists a path $\overline{P}$ in C_i which connects a_{i-1} and a_i and does not contain x. $S' = P_{i-1}^s \cup \overline{P} \cup P_i^t$ is a path connecting s and t. Since S' and D are vertex-disjoint and D has a degree 3 vertex, $G - V(S')$ has a degree 3 vertex, contradicting Lemma 9.

Thus, we conclude that $|V(D \cap S)| \neq 1$ and the claim holds. □

We prove by a sequence of claims that $\mathcal{C}$ and $\mathcal{A}$ satisfy (f) in Condition 2. It is clear that $P \in \mathcal{D}'$ has exactly one connection point. We denote the connection point by $c(P)$.

Claim 15: For $P \in \mathcal{D}'$, there exists a unique $C_i \in \mathcal{H}$ $(1 \leqq i \leqq m)$ such that $c(P) \in V(C_i)$. Moreover, $(c(P), a_{i-1}) \in E(C_i)$ or $(c(P), a_i) \in E(C_i)$.

Proof: Since $\Delta(G) = 3$, it is clear that for $P \in \mathcal{D}'$, there exists a unique $C_i \in \mathcal{H}$ $(1 \leqq i \leqq m)$ such that $c(P) \in V(C_i)$. We show that $(c(P), a_{i-1}) \in E(C_i)$ or $(c(P), a_i) \in E(C_i)$. We prove this by contradiction. Assume that $(c(P), a_{i-1}) \notin E(C_i)$ and $(c(P), a_i) \notin E(C_i)$. $c(P)$ is neither a_{i-1} nor a_i from Claim 11 and the assumption that $\Delta(G) = 3$. Thus, neither a_{i-1} nor a_i is contained in $N_G(c(P)) \cup \{c(P)\}$. Since C_i is 2-connected outer planar graph with $\Delta(G) = 3$, $c(P)$ is incident to just two outer edges of C_i and to exactly one edge of P. Thus, there exists a path $\overline{P}$ in C_i which connects a_{i-1} and a_i and does not contain a vertex incident to the two outer edges. $S' = P^s_{i-1} \cup \overline{P} \cup P^t_i$ is a path connecting s and t. Since S' has no vertex adjacent to $c(P)$, $G - V(S')$ has $c(P)$ with degree 3, contradicting Lemma 9. □

Claim 16: For distinct $P_1, P_2 \in \mathcal{D}'$, $c(P_1) \neq c(P_2)$.

Proof: Each $c(P_i)$ $(i = 1,2)$ is contained in a 2-connected component of G by Claim 15. If $c(P_1) = c(P_2)$ then $\deg_G(c(P_i)) \geq 4$ $(i = 1,2)$, contradicting the assumption that $\Delta(G) = 3$. □

Claim 17: Suppose that $C_i \in \mathcal{H}$ $(1 \leq i \leq m)$. If there exist distinct $P_1, P_2 \in \mathcal{D}'$ such that both $c(P_1)$ and $c(P_2)$ are adjacent to $a \in \{a_{i-1}, a_i\}$, then $c(P_1)$ or $c(P_2)$ is adjacent to $a' \in \{a_{i-1}, a_i\} - \{a\}$.

Proof: We show the claim by contradiction. Assume that there exist distinct $P_1, P_2 \in \mathcal{D}'$ such that both $c(P_1)$ and $c(P_2)$ are adjacent to $a \in \{a_{i-1}, a_i\}$ and that neither $c(P_1)$ nor $c(P_2)$ is adjacent to $a' \in \{a_{i-1}, a_i\} - \{a\}$. Let L be the subgraph of G induced by all the outer edges of C_i. Suppose that $N_L(a') = \{u, v\}$. It follows from the assumption and Claims 15 and 16 that a, a', u, v, $c(P_1)$, and $c(P_2)$ are distinct vertices.

If there exists an edge $e \in E(G) - E(C_i)$ incident to a', then M_3 shown in Fig. 1 is a minor of the subgraph $L \cup P_1 \cup P_2 \cup G[\{e\}]$ of G, i.e. $ppw(G) > 2$. This means that $\Gamma_G(a') - E(C_i) = \emptyset$ and that the proper-pathwidth of the graph G' obtained from G by adding an additional vertex x and by joining a' and x by an additional edge is more than 2. If $a' = a_j$ $(1 \leq j < m)$ then $\Gamma_G(a') - E(C_i) \neq \emptyset$ clearly. Thus we have that $a' = a_0 (= s)$ or $a' = a_m (= t)$. Let $\mathcal{X}' = (\{x, s\}) + \mathcal{X}$ if $a' = s$, $\mathcal{X}' = \mathcal{X} + (\{t, x\})$ otherwise. It is not difficult to see that $\mathcal{X}'$ is a proper-path-decomposition of G' and that the width of $\mathcal{X}'$ is 2. This means that $ppw(G') = 2$, a contradiction. □

Claim 18: For $C_i \in \mathcal{H}$ $(1 \leq i \leq m)$, $|\{P \in \mathcal{D}' \mid c(P) \in V(C_i)\}| \leq 2$.

Proof: We show the claim by contradiction. Assume that there exist distinct $P_1, P_2, P_3 \in \mathcal{D}'$ such that $\{c(P_1), c(P_2), c(P_3)\} \subseteq V(C_i)$. Let L be the subgraph of G induced by all the outer edges of C_i. Moreover, let G' be the graph obtained from G by adding additional vertices x and y and edges (x, s) and (y, t). Notice that there exist distinct edges $e \in \Gamma_{G'}(a_{i-1}) - E(C_i)$ and $e' \in \Gamma_{G'}(a_i) - E(C_i)$. As shown in the proof of Claim 15, $\{c(P_1), c(P_2), c(P_3)\} \cap \{a_{i-1}, a_i\} = \emptyset$. Thus it follows from Claim 16 that $c(P_1)$, $c(P_2)$, $c(P_3)$, a_{i-1}, and a_i are distinct vertices. Therefore, M_2 shown in Fig. 1 is a minor of the subgraph $L \cup P_1 \cup P_2 \cup P_3 \cup G'[\{e, e'\}]$ of G', i.e. $ppw(G') > 2$. However, it is not difficult to see that $\mathcal{X}' = (\{x, s\}) + \mathcal{X} + (\{t, y\})$ is a proper-path-decomposition of G' and that the width of $\mathcal{X}'$ is 2. Thus we have $ppw(G') = 2$, a contradiction. □

Claim 19: $\mathcal{C}$ and $\mathcal{A}$ satisfy (f) in Condition 2.

Proof: It follows from Claim 15 that there exists a mapping f satisfying the statement $(*)$ in Condition 2. By Claims 17 and 18, f can easily be reconstructed so that it is a one-to-one mapping satisfying $(*)$. □

Thus, $\mathcal{C}$ and $\mathcal{A}$ satisfy Condition 2. Therefore, the proof of necessity for Theorem 1 is completed.

Proof of Sufficiency for Theorem 1

We next show the sufficiency. Assume that G has a sequence $\mathcal{C} = (C_1, C_2, \ldots, C_m)$ of components in $\mathcal{D}$ and a sequence $\mathcal{A} = (a_0, a_1, \ldots, a_m)$ of vertices of G such that Condition 2 is satisfied. If $C_1 \in \mathcal{T}$ and $\deg_G(a_0) = 2$ then we can easily find a vertex $a'_0 \in V(C_1)$ such that $\deg_G(a'_0) = 1$ and that the path connecting a'_0 and a_1 is a 2-spine of C_1. Moreover, $\mathcal{C}$ and the sequence $(a'_0, a_1, \ldots, a_m)$ satisfy Condition 2. Thus, we assume without loss of generality that, if $C_1 \in \mathcal{T}$ then $\deg_G(a_0) = 1$. Similarly, we assume without loss of generality that, if $C_m \in \mathcal{T}$ then $\deg_G(a_m) = 1$.

For $C_i \in \mathcal{H}$ $(1 \leq i \leq m)$, we define that e^0_i and e^1_i are distinct edges of C_i incident to a_i and a_{i-1}, respectively, such that if there exists $P \in \mathcal{D}'$ with $f(P) = (i, j)$ then $e^j_i = (a_{i-j}, c(P))$ $(j = 0, 1)$. The following claim shows that e^0_i and e^1_i satisfy Condition 3 for $C_i \in \mathcal{H}$.

Claim 20: For $C_i \in \mathcal{H}$, e^0_i and e^1_i are outer edges of C_i. Moreover, they are contained in distinct end-regions if C_i has two end-regions.

Proof: The claim is immediate if C_i has a single end-region. Thus, we assume C_i has two end-regions. Since (b) in Condition 2 is satisfied and $\Delta(G) = 3$, we have that $\deg_{C_i}(a_{i-1}) = \deg_{C_i}(a_i) = 2$. Thus, two edges incident to $a \in \{a_{i-1}, a_i\}$ are outer edges contained in a same region. Moreover, since (d) in Condition 2 is satisfied, a_{i-1} and a_i are contained in distinct end-regions. Therefore, $\Gamma_{C_i}(a_{i-1})$ and $\Gamma_{C_i}(a_i)$ are subsets of edges of distinct end-regions. Since $e^j_i \in \Gamma_{C_i}(a_{i-j})$ $(j = 0, 1)$, the claim holds. □

We show that the sequence $\mathcal{X} = (X_1, \ldots, X_r)$ of subsets of $V(G)$ defined as follows is a 2-proper-path-decomposition of G.

$$\mathcal{X} = \sum_{1 \leq i \leq m} \mathcal{L}^i + \mathcal{Y}^i + \mathcal{R}^i, \text{ where for } 1 \leq i \leq m,$$

$$\mathcal{Y}^i = \begin{cases} \texttt{PPD\_TREE}(C_i, \text{path } (a_{i-1}, \ldots, a_i)) & \\ \qquad\qquad\qquad\qquad \text{if } C_i \in \mathcal{T} \cup \mathcal{P} \\ \texttt{PPD\_2CG}(C_i, e^1_i, e^0_i) \quad \text{if } C_i \in \mathcal{H} \end{cases}$$

$$\mathcal{L}^i = \begin{cases} \texttt{PPD\_PATH}(P = (p_0, \ldots, c(P))) \cup \{a_{i-1}\} \\ \qquad \text{if } \exists P \in \mathcal{D}' \text{ with } f(P) = (i, 1) \\ nul \qquad \text{otherwise} \end{cases}$$

$$\mathcal{R}^i = \begin{cases} \texttt{PPD\_PATH}(P = (c(P), \ldots, p_l)) \cup \{a_i\} \\ \qquad \text{if } \exists P \in \mathcal{D}' \text{ with } f(P) = (i, 0) \\ nul \qquad \text{otherwise} \end{cases}$$

$\mathcal{X}$ satisfies (a), (b), and (c) in Condition 1 by definition. Moreover, every element of $\mathcal{X}$ contains at most three vertices of G. Thus, it suffices to show that $\mathcal{X}$ satisfies (d) and (e) in Condition 1. By the definition of PPD_PATH and Corollaries 5 and 8, we can observe the following claim.

Claim 21:

1. For $1 \leqq i \leqq m$, $v \in V(C_i) - (\{a_{i-1}, a_i\} \cup \{c(P) \mid P \in \mathcal{D}'\})$ appears in consecutive elements of $\mathcal{Y}^i$.
2. For $P \in \mathcal{D}'$, $v \in V(P) - \{c(P)\}$ appears in at most two consecutive elements of $\mathcal{X}$.
3. For $0 \leqq i \leqq m$, a_i appears consecutive elements of $\mathcal{Y}^i + \mathcal{R}^i + \mathcal{L}^{i+1} + \mathcal{Y}^{i+1}$, where $\mathcal{Y}^0 = \mathcal{R}^0 = \mathcal{Y}^{m+1} = \mathcal{L}^{m+1} = nul$.
4. For $P \in \mathcal{D}'$ with $f(P) = (i, 1)$, $c(P)$ appears in the tail of $\mathcal{L}^i$ and in consecutive elements of $\mathcal{Y}^i$ including its head.
5. For $P \in \mathcal{D}'$ with $f(P) = (i, 0)$, $c(P)$ appears in the head of $\mathcal{R}^i$ and in consecutive elements of $\mathcal{Y}^i$ including its tail.

□

It follows from Claim 21 that every vertex in G appears in consecutive elements of $\mathcal{X}$. Thus, $\mathcal{X}$ satisfies (d) in Condition 1.

It remains to show that $\mathcal{X}$ satisfies (e) in Condition 1. If $X_a \cap X_c = \emptyset$ for all a and c with $1 < a+1 \leqq c-1 < r$, then this is immediate. Thus, we assume that there exist a and c with $1 < a+1 \leqq c-1 < r$ such that $X_a \cap X_c \neq \emptyset$. For $1 \leqq i \leqq m$, $\mathcal{Y}^i$ is a proper-path-decomposition of C_i. Thus, we have that $|X_a \cap X_c| \leqq |X_b| - 2$ for any b with $a < b < c$ if there exists i with $1 \leqq i \leqq m$ such that both X_a and X_c are elements of $\mathcal{Y}^i$. Therefore, we assume that there exists no i with $1 \leqq i \leqq m$ such that both X_a and X_c are elements of $\mathcal{Y}^i$. It follows from assumption and Claim 21 that $X_a \cap X_c$ contains at most one vertex in $\mathcal{A}$ and at most one vertex in $\{c(P) \mid P \in \mathcal{D}'\}$.

Claim 22: $|X_a \cap X_c| = 1$.

Proof: It suffices to show that both a_i $(0 \leqq i \leqq m)$ and $c(P)$ are not contained in $X_a \cap X_c$. We prove this by contradiction. Assume that there exist i $(0 \leqq i \leqq m)$ and $P \in \mathcal{D}'$ such that $\{a_i, c(P)\} \subseteqq X_a \cap X_c$. By Claim 21 and the assumption that no $\mathcal{Y}^i$ $(1 \leqq i \leqq m)$ contains both X_a and X_c, we have that $f(P) = (i, 0)$ or $f(P) = (i+1, 1)$. We may assume without loss of generality that $f(P) = (i, 0)$. Then, both X_a and X_c are elements of $\mathcal{Y}^i$ + (the head of $\mathcal{R}^i$). Suppose that $\mathcal{Y}^i = (Y_1^i, \ldots, Y_r^i)$. Since $c - a \geqq 2$, we have that $X_a \neq Y_r^i$. Thus, there exists j with $1 \leqq j < r$ such that $\{a_i, c(P)\} \subseteqq X_a = Y_j^i$. However, this is impossible since $(a_i, c(P)) = e_i^0 \in E(G[Y_r^i]) - E(G[\bigcup_{1 \leqq j < r} Y_j^i])$ by Corollary 8. □

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Procedure PPD_GENERAL ( G )
[ Input:  a connected graph G with Δ(G) ≦ 3;                    ]
[ Output: a 2-proper-path-decomposition of G;                   ]
  1. let H, T, and P be the sets of 2-connected components,
     tree components, and path components of G, respectively;
  2. D := H ∪ T ∪ P;
  3. find a sequence C = (C_1, C_2, ..., C_m) of components in D
     and a sequence A = (a_0, a_1, ..., a_m) of vertices of G such
     that Condition 2 and the following conditions are satisfied:
          deg_G(a_0) = 1 if C_1 ∈ T;
          deg_G(a_m) = 1 if C_m ∈ T;
  4. if C and A do not exist then reject ;
  5. D' := D − {C_i | 1 ≦ i ≦ m};
  6. for each C_i ∈ H do
       a. find distinct edges e_i^0 ∈ Γ_{C_i}(a_i) and e_i^1 ∈ Γ_{C_i}(a_{i−1})
          such that, if there exists P ∈ D' with f(P) = (i, j)
          then e_i^j = (a_{i−j}, c(P)) (j = 0, 1);
     endfor ;
  7. for i = 1 to m do
       a. if C_i ∈ T ∪ P then
          Y^i := PPD_TREE(C_i, path (a_{i−1}, ..., a_i));
          else Y^i := PPD_2CG(C_i, e_i^1, e_i^0);
       b. if ∃P ∈ D' with f(P) = (i, 1) then
          L^i := PPD_PATH(P = (p_0, ..., c(P))) ∪ {a_{i−1}};
          else L^i := nul;
       c. if ∃P ∈ D' with f(P) = (i, 0) then
          R^i := PPD_PATH(P = (c(P), ..., p_l)) ∪ {a_i};
          else R^i := nul;
     endfor ;
  8. return Σ_{1≦i≦m} L^i + Y^i + R^i;
End
```

Fig. 5 Algorithm for constructing a 2-proper-path-decomposition of a general graph.

Claim 23: $|X_b| = 3$ for any b with $a < b < c$.

Proof: Let b be any integer such that $a < b < c$. If there exists i $(1 \leqq i \leqq m)$ such that X_b is an element of $\mathcal{Y}^i$ and that $C_i \in \mathcal{H}$, then $|X_b| = 3$ by Corollary 8. If there exists i $(1 \leqq i \leqq m)$ such that X_b is an element of $\mathcal{L}^i$ or $\mathcal{R}^i$, then $|X_b| = 3$ by the definition of PPD_PATH and by the fact that $|V(P)| \geqq 2$ for any $P \in \mathcal{D}'$. Thus, it suffices to show that X_b is not an element of $\mathcal{Y}^i$ such that $C_i \in \mathcal{T} \cup \mathcal{P}$. We prove this by contradiction. Assume that X_b is an element of $\mathcal{Y}^i$ $(1 \leqq i \leqq m)$ such that $C_i \in \mathcal{T} \cup \mathcal{P}$. It follows from the assumption and Claim 22 that either $X_a \cap X_c = \{a_{i-1}\}$ or $X_a \cap X_c = \{a_i\}$. We assume without loss of generality that $X_a \cap X_c = \{a_i\}$. Since X_b is an element of $\mathcal{Y}^i$, X_a is an element of $\mathcal{Y}^i$ except the tail. This means that a_i is contained in an element of $\mathcal{Y}^i$ except the tail. However, this is impossible since a_i is an end-vertex of 2-spine of C_i and a_i appears only in the tail of $\mathcal{Y}^i$ by Corollary 5. □

It follows from Claims 22 and 23 that $|X_a \cap X_c| -$

$|X_b| = 3 - 2 = 1$ for $a < b < c$. Thus, $\mathcal{X}$ satisfies (e) in Condition 1.

Therefore, $\mathcal{X}$ is a 2-proper-path-decomposition of G and the proof of sufficiency for Theorem 1 is completed.

We describe in Fig. 5 Algorithm PPD_GENERAL based on Theorem 1. It is well-known that we can find all blocks of a graph in linear time. Moreover, we can determine if a given graph is outer planar in linear time [4]. To find a_0 and a_m in step 3, we need an algorithm to find a 2-spine of a binary tree, which has not been described yet. Although the details are not mentioned here, this can be done in linear time by using a simple postorder searching and the algorithm in [8], which outputs, for a rooted binary tree, the proper-pathwidth of every subtree rooted at a vertex. The other operations in PPD_GENERAL clearly executed in linear time.

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