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Fault-Tolerant Hypercubes with Small Degree

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SUMMARY For a given N-vertex graph H, a graph G obtained from H by adding t vertices and some edges is called a t-FT (t-fault-tolerant) graph for H if even after deleting any t vertices from G, the remaining graph contains H as a subgraph. For the n-dimensional cube $Q(n)$ with N vertices, a t-FT graph with an optimal number $O(tN+t^2)$ of added edges and maximum degree of $O(N+t)$, and a t-FT graph with $O(tN\log N)$ added edges and maximum degree of $O(t\log N)$ have been known. In this paper, we introduce some t-FT graphs for $Q(n)$ with an optimal number $O(tN+t^2)$ of added edges and small maximum degree. In particular, we show a t-FT graph for $Q(n)$ with $2ctN+ct^2\left(\frac{\log N}{c}\right)^c$ added edges and maximum degree of $O(\frac{N}{\log^{c/2} N})+4ct$.

key words: *hypercubes, fault-tolerant graphs, maximum degree, multi-processor systems, interconnection networks*

1. Introduction

The hypercube is one of the well-known topologies for interconnection networks of multiprocessor systems. However, even a small number of faulty processors and/or communication links can seriously affect the performance of hypercube machines. We show a fault-tolerant architecture for hypercubes in which spare processors and communication links are added so that the architecture contains a fault-free hypercube even in the presence of faults. We optimize the cost of the fault-tolerant architecture by adding exactly t spare processors, while tolerating up to t processor and/or link faults, and minimizing the number of spare links and the maximum number of links per processor. This architecture guarantees that any algorithm designed for the hypercube will run with no slowdown in the presence of t or fewer faults, regardless of their distribution.

Our approach is based on a graph model initiated by Hayes [20], in which each vertex and edge represent a processor and communication link, respectively. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. Let $\Delta(G)$ denote the maximum degree of a vertex in G. For any $S\subseteq V(G)$, $G-S$ is the graph obtained from G by deleting the vertices of S together with the edges incident to the vertices in S. Let t be a positive integer. A graph G is called a t-FT (t-fault-tolerant) graph for a graph H if $G-F$ contains H as a subgraph for every $F\subseteq V(G)$ with $|F|\leq t$. Our problem is to construct a t-FT graph G for the n-dimensional cube $Q(n)$ with $N=2^n$ vertices such that $|V(G)|$, $|E(G)|$, and $\Delta(G)$ are minimized.

$G\vee H$ is the graph obtained from graphs G and H by connecting each vertex of G and each vertex of H by an edge. It is easy to see that $H\vee K_t$ is a t-FT graph for any graph H, where K_t is the complete graph with t vertices. $H\vee K_t$ is obtained from an N-vertex graph H by adding t vertices and $tN+\frac{1}{2}t(t-1)$ edges.

Since the degree of every vertex of $Q(n)$ is $\log N$, the minimum degree of a vertex in an $(N+t)$-vertex t-FT graph for $Q(n)$ is at least $\log N+t$, and so at least $\Omega(tN+t^2)$ edges must be added to $Q(n)$ in order to construct an $(N+t)$-vertex t-FT graph for $Q(n)$. Thus, $Q(n)\vee K_t$ is an optimal t-FT graph for $Q(n)$ in the sense that the number of edges added to $Q(n)$ is optimal to within a constant factor. However, $\Delta(Q(n)\vee K_t)=N+t-1$, and $Q(n)\vee K_t$ is not practical at all.

Bruck, Cypher, and Ho [6] proposed another construction of t-FT graph for $Q(n)$. Their t-FT graph for $Q(n)$ has a small maximum degree of $O(t\log N)$. However, their t-FT graph is constructed from $Q(n)$ by adding $\Omega(tN\log N)$ edges, which is a relatively large number.

This paper proposes three t-FT graphs for $Q(n)$ with $O(tN)$ added edges and relatively small maximum degrees. A key idea of our constructions is to partition the vertices of $Q(n)$ according to the distribution of 1s in the label of a vertex. In Sect. 3, we show a naive construction of a t-FT graph for $Q(n)$ with $2tN+t^2$ added edges and maximum degree of $O(N/\sqrt{\log N})+3t$. The construction is based on a partition of the vertices of $Q(n)$ according to the Hamming weight of the label of a vertex. Based on a refinement of the partition above, we give in Sect. 4 an improved construction of a t-FT graph for $Q(n)$ with $4tN+2t^2$ added edges and maximum degree of $O(N/\log N)+5t$. Finally, based on a further refinement of the partition used in Sect. 4, we present in Sect. 5 a sophisticated construction of a t-FT graph for $Q(n)$ with $2ctN+ct^2(\log N/c)^c$ added edges and maximum degree of $O(N/\log^{c/2} N)+4ct$ for any fixed integer c.

It should be noted that, in the edge-fault case, a near optimal fault-tolerant hypercube has been con-

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structed. It is shown in [31] that we can construct a graph G from $Q(n)$ by adding $O(tN\log(\log N/t+c))$ edges such that $\Delta(G)=\log N+O(t\log(\log N/t+c))$ and even after deleting any t edges from G, the remaining graph contains $Q(n)$ as a subgraph.

2. Preliminaries

The n-cube (n-dimensional cube), denoted by $Q(n)$, is defined as follows: $V(Q(n))=\{0,1\}^n$; $E(Q(n))=\{(u,v)|\ u,v\in V(Q(n)),\ w(u\oplus v)=1\}$, where $\oplus$ denotes bit-wise addition modulo 2 and $w(x)$ is the Hamming weight of binary vector x, that is the number of 1's which x contains. It is easy to see that $|V(Q(n))|=2^n$. Since each vertex of $Q(n)$ has degree n, $|E(Q(n))|=n2^{n-1}$. A graph G is called a hypercube if G is isomorphic to $Q(n)$ for some n.

For a t-FT graph G for $Q(n)$, define $\Lambda(G)=|E(G)|-|E(Q(n))|=|E(G)|-n2^{n-1}$. That is, $\Lambda(G)$ is the number of edges added to $Q(n)$ in order to construct G.

Throughout the paper, let $[n]=\{0,1,2,\dots,n-1\}$ and $[n]^+=\{1,2,\dots,n-1\}$, and let $N=2^n$.

3. t-FT Graph $G^1(n)$ for $Q(n)$ with $\Delta(G^1(n))=O(N/\log^{1/2}N)$ and $\Lambda(G^1(n))=O(tN)$

For any k odd, define ϕ_k as the mapping from $[k]$ to $[k]$ such that $\phi_k(i)=(2i)\bmod k$.

Lemma 1: ϕ_k is a bijection. In particular, $\phi_k(0)=\phi_k^{-1}(0)=0$.

Proof: Suppose that $\phi_k(i)=\phi_k(j)$ for some $i,j\in[k]$. Then, $(2(i-j))\bmod k=0$. Since k is odd, we have $(i-j)\bmod k=0$. Since $|i-j|\in[k]$, we conclude that $i-j=0$, that is $i=j$. Thus, ϕ_k is a one-to-one mapping, and hence a bijection.

Since $\phi_k(0)=0$, $\phi_k^{-1}(0)=0$. □

Lemma 2: $(\phi_k^{-1}((i+2)\bmod k)-1)\bmod k=\phi_k^{-1}(i)$ for any $i\in[k]$.

Proof: Let $j=(i+2)\bmod k$. Since

$$\begin{aligned}\phi_k((\phi_k^{-1}(j)-1)\bmod k)&=((2\phi_k^{-1}(j)-2)\bmod k)\bmod k\\&=(2\phi_k^{-1}(j)\bmod k-2)\bmod k\\&=(j-2)\bmod k\\&=i,\end{aligned}$$

we have $(\phi_k^{-1}((i+2)\bmod k)-1)\bmod k=\phi_k^{-1}(i)$. □

Let $k=n$ if n is odd, and $k=n-1$ otherwise. Note that k is odd. Define that $V_i=\{v\in V(Q(n))|w(v)\bmod k=i\}$ for any $i\in[k]$. It is easy to see that $(V_0,V_1,\dots,V_{k-1})$ is a partition of $V(Q(n))$. Note that $\min_{i\in[k]^+}|V_i|=n$ if n is odd, and $\min_{i\in[k]^+}|V_i|=n+1$ otherwise.

Let $n\geqq 3$ and $t\leqq n$. For any $i\in[k]^+$, let $S_i\subset V_i$ such that $|S_i|=t$. $G^1(n)$ is the graph defined as follows:

$$\begin{aligned}V(G^1(n))&=V(Q(n))\cup S_0;\\E(G^1(n))&=E(Q(n))\\&\quad\cup\bigcup_{i=0}^{k-1}\{(u,v)|u\in V_i,v\in S_{(i+1)\bmod k}\}\\&\quad\cup\bigcup_{i=0}^{k-1}\{(u,v)|u\in V_i,v\in S_{(i+3)\bmod k}\}\\&\quad\cup\{(u,v)|u\in S_0,v\in S_1\},\end{aligned}$$

where S_0 is the set of t vertices added to $Q(n)$.

Lemma 3: $G^1(n)$ is a t-FT graph for $Q(n)$.

Proof: Let F be any subset of $V(G^1(n))$ such that $|F|\leqq t$. Let $F_i=V_i\cap F$ and $t_i=|F_i|$ for any $i\in[k]$, and let $F_k=S_0\cap F$ and $t_k=|F_k|$. Since $(F_0,F_1,\dots,F_k)$ is a partition of F, $|F|=\sum_{i=0}^{k}t_i\leqq t$. Since $\phi_k(\phi_k^{-1}(i))=i$ for any $i\in[k]$, we have, by Lemma 1,

$$\sum_{j=0}^{\phi_k^{-1}(i)-1}t_{\phi_k(j)}\leqq t-t_i$$

for any $i\in[k]^+$ and

$$\sum_{j=0}^{k-1}t_{\phi_k(j)}=\sum_{j=0}^{k-1}t_j\leqq t-t_k.$$

Thus, there exists $A_i\subset S_i-F$ such that

$$|A_i|=\begin{cases}\displaystyle\sum_{j=0}^{k-1}t_j & \text{if } i=0,\\ \displaystyle\sum_{j=0}^{\phi_k^{-1}(i)-1}t_{\phi_k(j)} & \text{if } i\in[k]^+.\end{cases}$$

By Lemma 2, we have $|F_i\cup A_i|=|F_i|+|A_i|=|A_{(i+2)\bmod k}|=\sum_{j=0}^{\phi_k^{-1}(i)}t_{\phi_k(j)}$ for any $i\in[k]^+$, and $|F_0|=|A_2|=t_0$. Thus, there exist bijections

$$\varphi_i:\begin{cases}F_0\to A_2 & \text{if } i=0,\\ F_i\cup A_i\to A_{(i+2)\bmod k} & \text{if } i\in[k]^+.\end{cases}$$

Define the mapping φ from $V(Q(n))$ to $V(G^1(n)-F)$ as follows:

$$\varphi(v)=\begin{cases}v & \text{if } v\notin F\cup A,\\ \varphi_0(v) & \text{if } v\in F_0,\\ \varphi_i(v) & \text{if } v\in F_i\cup A_i,\ i\in[k]^+,\end{cases}$$

where $A=\bigcup_{i=1}^{k-1}A_i$. It is easy to see that φ is a one-to-one mapping.

Now, we will show that $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$ for any $(u,v) \in E(Q(n))$. We assume without loss of generality that $u \in V_i$ and $v \in V_{(i+1) \bmod k}$ for some $i \in [k]$. There are four cases as follows.

Case 1 $u,v \notin F \cup A$: Since $\varphi(u)=u$ and $\varphi(v)=v$, we have $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$.

Case 2 $u \notin F \cup A, v \in F \cup A$: Since $\varphi(u)=u \in V_i$ and $\varphi(v)=\varphi_{(i+1) \bmod k}(v) \in A_{(i+3) \bmod k} \subseteq S_{(i+3) \bmod k}$, we have $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$.

Case 3 $u \in F \cup A, v \notin F \cup A$: Since $\varphi(u)=\varphi_i(u) \in A_{(i+2) \bmod k} \subseteq S_{(i+2) \bmod k}$ and $\varphi(v)=v \in V_{(i+1) \bmod k}$, we have $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$.

Case 4 $u,v \in F \cup A$: If $i \neq k-2$ then $\varphi(u)=\varphi_i(u) \in A_{(i+2) \bmod k} \subseteq V_{(i+2) \bmod k}$ and $\varphi(v)=\varphi_{(i+1) \bmod k}(v) \in A_{(i+3) \bmod k} \subseteq S_{(i+3) \bmod k}$. Thus, $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$. If $i=k-2$ then $\varphi(u)=\varphi_{k-2}(u) \in A_0 \subseteq S_0$ and $\varphi(v)=\varphi_{k-1}(v) \in A_1 \subseteq S_1$. Thus, $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$.

Thus, $(\varphi(u),\varphi(v)) \in E(G^1(n)-F)$ for any $(u,v) \in E(Q(n))$, and so $G^1(n)-F$ contains $Q(n)$ as a subgraph. Hence $G^1(n)$ is a t-FT graph for $Q(n)$. □

Now we estimate the maximum degree of $G^1(n)$ and the number of edges added to $Q(n)$ to construct $G^1(n)$. We need the following lemma.

Lemma 4: [16] $\binom{n}{\lfloor n/2 \rfloor} = \Theta\left(\frac{2^n}{\sqrt{n}}\right)$.

Lemma 5: $\Delta(G^1(n)) = O(2^n/\sqrt{n})+3t$.

Proof: Let $deg_1(v)$ denote the degree of $v \in V(G^1(n))$. There are five cases as follows.

Case 1 $v \in V_0$: $deg_1(v) \leqq n+|S_1|+|S_3| = n+2t$.

Case 2 $v \in S_0$: $deg_1(v) \leqq |V_{k-1}|+|V_{k-3}|+|S_1| \leqq \frac{1}{6}(n^3-3n^2+8n+6)+t$.

Case 3 $v \in V_i - S_i$, $i \in [k]^+$: $deg_1(v) \leqq n+|S_{(i+1) \bmod k}|+|S_{(i+3) \bmod k}| = n+2t$.

Case 4 $v \in S_1$: $deg_1(v) \leqq n+|V_0|+|V_{k-2}|+|S_2|+|S_4|+|S_0| \leqq \frac{1}{2}(n^2+3n+2)+3t$.

Case 5 $v \in S_i$, $i \neq 1$, $i \in [k]^+$: $deg_1(v) \leqq n+|V_{(i-1) \bmod k}|+|V_{(i-3) \bmod k}|+|S_{(i+1) \bmod k}|+|S_{(i+3) \bmod k}| \leqq n+2\max_{j \in [k]}|V_j|+2t$.

Since, by Lemma 4,

$$\max_{j \in [k]}|V_j| = \binom{n}{\lfloor n/2 \rfloor} = \Theta\left(\frac{2^n}{\sqrt{n}}\right),$$

we conclude that $\Delta(G^1(n)) = O(2^n/\sqrt{n})+3t$. □

Lemma 6: $\Lambda(G^1(n)) \leqq t2^{n+1}+t^2$.

Proof: Since $|S_i|=t$ for any $i \in [k]$, we have

$$\Lambda(G^1(n)) \leqq 2t\sum_{j \in [k]}|V_j|+t^2 = t2^{n+1}+t^2.$$

□

By summarizing Lemmas 3, 5, and 6, we have the following theorem.

Theorem 1: Let $n \geqq 3$ and $t \leqq n$. $G^1(n)$ is a t-FT graph for $Q(n)$ with $2tN+t^2$ added edges and maximum degree of $O(N/\sqrt{\log N})+3t$. □

Theorem 1 can be generalized for larger t. Let α be an integer greater than 2 and $k = 2\lceil n/(2\alpha)\rceil - 1$. The following theorem can be proved by a similar argument as the proof of Theorem 1, but we omit the details here.

Theorem 2: Let α be a integer greater than 2 and let $n \geqq 2\alpha+1$ and let $\lambda = \frac{1}{2}(1-\frac{1}{\alpha})$. If $t \leqq \binom{n}{\lfloor \lambda n \rfloor}$, then we can construct a t-FT graph for $Q(n)$ with $2tN+t^2$ added edges and maximum degree of $O(N/\sqrt{\log N})+3t$. □

4. t-FT Graph $G^2(n)$ for $Q(n)$ with $\Delta(G^2(n)) = O(N/\log N)$ and $\Lambda(G^2(n)) = O(tN)$

Throughout this section, let $\mathbf{o}$ denote $(0,0)$, and let $\mathbb{L}_k = [k+1] \times [k]$ and $\mathbb{L}_k^+ = [k+1] \times [k] - \{\mathbf{o}\}$. For any $\boldsymbol{i} = (i_1,i_2) \in \mathbb{L}_k$ and $\boldsymbol{j} = (j_1,j_2) \in \mathbb{L}_k$, define $\boldsymbol{i}+\boldsymbol{j} = ((i_1+j_1) \bmod (k+1), (i_2+j_2) \bmod k)$ and $\boldsymbol{i}-\boldsymbol{j} = ((i_1-j_1) \bmod (k+1), (i_2-j_2) \bmod k)$. For any k, define μ_k as the mapping from $[k(k+1)]$ to $\mathbb{L}_k$ such that $\mu_k(i) = (i \bmod (k+1), i \bmod k)$.

Lemma 7: μ_k is a bijection. In particular, $\mu_k(0) = \mathbf{o}$ and $\mu_k^{-1}(\mathbf{o}) = 0$.

Proof: Suppose that $\mu_k(i) = \mu_k(j)$ for some $i,j \in [k(k+1)]$. Then $((i-j) \bmod (k+1), (i-j) \bmod k) = \mathbf{o}$. Since k and $k+1$ are relatively prime, we have $(i-j) \bmod (k(k+1)) = 0$. Since $|i-j| \in [k(k+1)]$, we obtain $i-j=0$, that is $i=j$. Thus μ_k is a one-to-one mapping. Since $|[k(k+1)]| = |\mathbb{L}_k|$, we conclude that μ_k is a bijection.

Since $\mu_k(0) = \mathbf{o}$, $\mu_k^{-1}(\mathbf{o}) = 0$. □

Lemma 8: $(\mu_k^{-1}(\boldsymbol{i}+(1,1))-1) \bmod k(k+1) = \mu_k^{-1}(\boldsymbol{i})$ for any $\boldsymbol{i} \in \mathbb{L}_k$.

Proof: Let $\boldsymbol{i} = (i_1,i_2)$ and $\boldsymbol{j} = \boldsymbol{i}+(1,1)$. Since

$$\begin{aligned}
&\mu_k((\mu_k^{-1}(\boldsymbol{j})-1) \bmod k(k+1)) \\
&= ((\mu_k^{-1}(\boldsymbol{j})-1) \bmod (k+1), \\
&\qquad (\mu_k^{-1}(\boldsymbol{j})-1) \bmod k) \\
&= (((i_1+1) \bmod (k+1)-1) \bmod (k+1), \\
&\qquad ((i_2+1) \bmod k-1) \bmod k) \\
&= (i_1,i_2),
\end{aligned}$$

we have $(\mu_k^{-1}(\boldsymbol{i}+(1,1))-1) \bmod k(k+1) = \mu_k^{-1}(\boldsymbol{i})$. □

Let $k = (n-1)/2$ if n is odd, and $k = (n/2)-1$ otherwise. For any $(i,j) \in \mathbb{L}_k$, let

$$V_{(i,j)} = \left\{ v \in V(Q(n)) \;\middle|\; \begin{array}{c} w_u(v) \bmod (k+1) = i, \\ w_l(v) \bmod k = j \end{array} \right\},$$

where $w_u(v)$ and $w_l(v)$ are the numbers of 1s contained in $\lceil n/2 \rceil$ upper bits and $n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$ lower bits, respectively. Notice that $(V_{(0,0)}, V_{(0,1)}, \ldots, V_{(k,k-1)})$ is a partition of $V(Q(n))$. Note also that $\min_{\boldsymbol{i} \in \mathbb{L}_k^+} |V_{\boldsymbol{i}}| = n - 1$ if n is odd, and $\min_{\boldsymbol{i} \in \mathbb{L}_k^+} |V_{\boldsymbol{i}}| = n + 2$ otherwise.

Let $n \geqq 5$ and $t \leqq n - 1$. For any $\boldsymbol{i} \in \mathbb{L}_k^+$, let $S_{\boldsymbol{i}} \subset V_{\boldsymbol{i}}$ such that $|S_{\boldsymbol{i}}| = t$. $G^2(n)$ is the graph defined as follows:

$$\begin{aligned} V(G^2(n)) &= V(Q(n)) \cup S_{\mathbf{o}}; \\ E(G^2(n)) &= E(Q(n)) \\ &\cup \bigcup_{\boldsymbol{i} \in \mathbb{L}_k} \{(u,v) | u \in V_{\boldsymbol{i}}, v \in S_{\boldsymbol{i}+(1,0)}\} \\ &\cup \bigcup_{\boldsymbol{i} \in \mathbb{L}_k} \{(u,v) | u \in V_{\boldsymbol{i}}, v \in S_{\boldsymbol{i}+(0,1)}\} \\ &\cup \bigcup_{\boldsymbol{i} \in \mathbb{L}_k} \{(u,v) | u \in V_{\boldsymbol{i}}, v \in S_{\boldsymbol{i}+(2,1)}\} \\ &\cup \bigcup_{\boldsymbol{i} \in \mathbb{L}_k} \{(u,v) | u \in V_{\boldsymbol{i}}, v \in S_{\boldsymbol{i}+(1,2)}\} \\ &\cup \{(u,v) | u \in S_{\mathbf{o}}, v \in S_{(1,0)}\} \\ &\cup \{(u,v) | u \in S_{\mathbf{o}}, v \in S_{(0,1)}\}, \end{aligned}$$

where $S_{\mathbf{o}}$ is the set of t vertices added to $Q(n)$.

Lemma 9: $G^2(n)$ is a t-FT graph for $Q(n)$.

Proof: Let F be any subset of $V(G^2(n))$ such that $|F| \leqq t$. Let $F_{\boldsymbol{i}} = V_{\boldsymbol{i}} \cap F$ and $t_{\boldsymbol{i}} = |F_{\boldsymbol{i}}|$ for any $\boldsymbol{i} \in \mathbb{L}_k$, and let $F_{(k+1,k)} = S_{(0,0)} \cap F$ and $t_{(k+1,k)} = |F_{(k+1,k)}|$. Since $(F_{(0,0)}, F_{(0,1)}, \ldots, F_{(k,k-1)}, F_{(k+1,k)})$ is a partition of F, $|F| = \sum_{\boldsymbol{i} \in \mathbb{L}_k} t_{\boldsymbol{i}} + t_{(k+1,k)} \leqq t$. Since $\mu_k(\mu_k^{-1}(\boldsymbol{i})) = \boldsymbol{i}$ for any $\boldsymbol{i} \in \mathbb{L}_k$, we have, by Lemma 7,

$$\sum_{j=0}^{\mu_k^{-1}(\boldsymbol{i})-1} t_{\mu_k(j)} \leqq t - t_{\boldsymbol{i}}$$

for any $\boldsymbol{i} \in \mathbb{L}_k^+$ and

$$\sum_{j=0}^{k(k+1)-1} t_{\mu_k(j)} \leqq t - t_{(k+1,k)}.$$

Thus, for any $\boldsymbol{i} \in \mathbb{L}_k$, there exists $A_{\boldsymbol{i}} \subset S_{\boldsymbol{i}} - F$ such that

$$|A_{\boldsymbol{i}}| = \begin{cases} \displaystyle\sum_{j=0}^{k(k+1)-1} t_{\mu_k(j)} & \text{if } \boldsymbol{i} = \mathbf{o}, \\ \displaystyle\sum_{j=0}^{\mu_k^{-1}(\boldsymbol{i})-1} t_{\mu_k(j)} & \text{if } \boldsymbol{i} \in \mathbb{L}_k^+. \end{cases}$$

By Lemma 8, we have $|F_{\boldsymbol{i}} \cup A_{\boldsymbol{i}}| = |F_{\boldsymbol{i}}| + |A_{\boldsymbol{i}}| = |A_{\boldsymbol{i}+(1,1)}| = \sum_{j=0}^{\mu_k^{-1}(\boldsymbol{i})} t_{\mu_k(j)}$ for any $\boldsymbol{i} \in \mathbb{L}_k^+$, and $|F_{\mathbf{o}}| = |A_{(1,1)}| = t_{\mathbf{o}}$. Thus, there exist bijections

$$\nu_{\boldsymbol{i}} : \begin{cases} F_{\mathbf{o}} \rightarrow A_{(1,1)} & \text{if } \boldsymbol{i} = \mathbf{o}, \\ F_{\boldsymbol{i}} \cup A_{\boldsymbol{i}} \rightarrow A_{\boldsymbol{i}+(1,1)} & \text{if } \boldsymbol{i} \in \mathbb{L}_k^+. \end{cases}$$

Define the mapping ν from $V(Q(n))$ to $V(G^2(n) - F)$ as follows:

$$\nu(v) = \begin{cases} v & \text{if } v \notin F \cup A, \\ \nu_{\mathbf{o}}(v) & \text{if } v \in F_{\mathbf{o}}, \\ \nu_{\boldsymbol{i}}(v) & \text{if } v \in F_{\boldsymbol{i}} \cup A_{\boldsymbol{i}},\ \boldsymbol{i} \in \mathbb{L}_k^+, \end{cases}$$

where $A = \bigcup_{\boldsymbol{i} \in \mathbb{L}_k^+} A_{\boldsymbol{i}}$. It is easy to see that ν is a one-to-one mapping.

Now, we will show that $(\nu(u), \nu(v)) \in E(G^2(n) - F)$ for any $(u,v) \in E(Q(n))$. We may assume without loss of generality that $u \in V_{\boldsymbol{i}}$ and $v \in V_{\boldsymbol{i}+(1,0)}$ for some $\boldsymbol{i} \in \mathbb{L}_k$. There are four cases as follows.

Case 1 $u, v \notin F \cup A$: Since $\nu(u) = u$ and $\nu(v) = v$, we have $(\nu(u), \nu(v)) \in E(G^2(n) - F)$.

Case 2 $u \notin F \cup A$, $v \in F \cup A$: Since $\nu(u) = u \in V_{\boldsymbol{i}}$ and $\nu(v) = \nu_{\boldsymbol{i}+(1,0)}(v) \in A_{\boldsymbol{i}+(2,1)} \subset S_{\boldsymbol{i}+(2,1)}$, we have $(\nu(u), \nu(v)) \in E(G^2(n) - F)$.

Case 3 $u \in F \cup A$, $v \notin F \cup A$: Since $\nu(u) = \nu_{\boldsymbol{i}}(u) \in A_{\boldsymbol{i}+(1,1)} \subseteqq S_{\boldsymbol{i}+(1,1)}$ and $\nu(v) = v \in V_{\boldsymbol{i}+(1,0)}$, we have $(\nu(u), \nu(v)) \in E(G^2(n) - F)$.

Case 4 $u, v \in F \cup A$: If $\boldsymbol{i} \neq (k, k-1)$ then $\nu(u) = \nu_{\boldsymbol{i}}(u) \in A_{\boldsymbol{i}+(1,1)} \subseteqq V_{\boldsymbol{i}+(1,1)}$ and $\nu(v) = \nu_{\boldsymbol{i}+(1,0)}(v) \in A_{\boldsymbol{i}+(2,1)} \subseteqq S_{\boldsymbol{i}+(2,1)}$. Thus, $(\nu(u), \nu(v)) \in E(G^2(n) - F)$. If $(i_1, i_2) = (k, k-1)$ then $\nu(u) = \nu_{(k,k-1)}(u) \in A_{\mathbf{o}} \subseteqq S_{\mathbf{o}}$ and $\nu(v) = \nu_{(0,k-1)}(v) \in A_{(1,0)} \subseteqq S_{(1,0)}$. Thus, $(\nu(u), \nu(v)) \in E(G^2(n) - F)$.

Thus, $(\nu(u), \nu(v)) \in E(G^2(n) - F)$ for any $(u,v) \in E(Q(n))$, and so $G^2(n) - F$ contains $Q(n)$ as a subgraph. Hence $G^2(n)$ is a t-FT graph for $Q(n)$. □

Lemma 10: $\Delta(G^2(n)) = O(2^n/n) + 5t$.

Proof: Let $deg_2(v)$ denote the degree of $v \in V(G^2(n))$. There are five cases as follows.

Case 1 $v \in V_{\mathbf{o}}$: $deg_2(v) \leqq n + 4t$.

Case 2 $v \in S_{\mathbf{o}}$: $deg_2(v) \leqq 4 \max_{\boldsymbol{j} \in \mathbb{L}_k} |V_{\boldsymbol{j}}| + 2t$.

Case 3 $v \in V_{\boldsymbol{i}} - S_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{L}_k^+$: $deg_2(v) \leqq n + 4t$.

Case 4 $v \in S_{\boldsymbol{i}}$, $\boldsymbol{i} \in \mathbb{L}_k^+ - \{(0,1),(1,0)\}$: $deg_2(v) \leqq n + 4 \max_{\boldsymbol{j} \in \mathbb{L}_k} |V_{\boldsymbol{j}}| + 4t$.

Case 5 $v \in S_{(0,1)} \cup S_{(1,0)}$: $deg_2(v) \leqq n + 4 \max_{\boldsymbol{j} \in \mathbb{L}_k} |V_{\boldsymbol{j}}| + 5t$.

Since

$$\max_{\boldsymbol{j} \in \mathbb{L}_k} |V_{\boldsymbol{j}}| = \binom{m}{\lfloor m/2 \rfloor}^2$$

if $n = 2m$, and

$$\max_{j\in\mathbb{L}_k} |V_j| = \binom{m+1}{\lfloor (m+1)/2 \rfloor}\binom{m}{\lfloor m/2 \rfloor}$$

if $n = 2m+1$, we have, by Lemma 4,

$$\max_{j\in\mathbb{L}_k} |V_j| = \Theta\left(\frac{2^n}{n}\right),$$

and we conclude that $\Delta(G^2(n)) = O(2^n/n) + 5t$. □

Lemma 11: $\Lambda(G^2(n)) \leqq t2^{n+2} + 2t^2$.

Proof: Since $|S_j| = t$ for any $j \in \mathbb{L}_k$, we have

$$\Lambda(G^2(n)) \leqq 4t \sum_{j\in\mathbb{L}_k} |V_j| + 2t^2 = t2^{n+2} + 2t^2.$$

□

By summarizing Lemmas 9, 10, and 11, we have the following theorem.

Theorem 3: Let $n \geqq 5$ and $t \leqq n-1$. Then $G^2(n)$ is a t-FT graph for $Q(n)$ with $4tN + 2t^2$ added edges and maximum degree of $O(N/\log N) + 5t$. □

5. t-FT graph for $Q(n)$ with $\Delta(G^3(n)) = O(N/\log^{c/2}N) + 4ct$ and $\Lambda(G^3(n)) = 2ctN + ct^2(\log N/c)^c$

Let c be a fixed integer. Assume that $c|n$, and let $m = n/c \geqq 2$ and $M = m^c$. For any $i \in [M]$, let

$$V_i = \{v \in V(Q(n)) | \sum_{k=0}^{c-1} (w_k(v) \bmod m) m^k = i\},$$

where $w_k(v)$ is the number of 1s in the bit positions from the $(mk+1)$-st bit to the $m(k+1)$-st bit of v. Notice that $(V_0, \ldots, V_{M-1})$ is a partition of $V(Q(n))$. Note also that $\min_{i\in[M]^+} |V_i| = 2^{c-1}m = 2^{c-1}n/c$. For any $i \in [M]$, let $Neib(i) = \{j | u \in V_i$ and $v \in V_j$ for some $(u,v) \in E(Q(n))\}$. Since $m \geqq 2$, if $(u,v) \in E(Q(n))$ then there exists $k_1 \in [c]$ such that $w_{k_1}(v) = w_{k_1}(u) \pm 1$ and $w_k(v) = w_k(u)$ for every $k \neq k_1$. Thus, $|Neib(i)| \leqq 2c$ for any $i \in [M]$.

Let $t \leqq 2^{c-1}n/c$. For any $i \in [M]^+$, let $S_i \subset V_i$ such that $|S_i| = t$. $G^3(n)$ is the graph defined as follows:

$$\begin{aligned} V(G^3(n)) &= V(Q(n)) \cup S_0; \\ E(G^3(n)) &= E(Q(n)) \\ &\cup \bigcup_{i=0}^{M-1} \bigcup_{j\in Neib(i)} \{(u,v) | u \in V_i, \\ &\qquad v \in S_{(j+1)\bmod M}\} \\ &\cup \bigcup_{i=0}^{M-1} \bigcup_{j\in Neib(i)} \{(u,v) | u \in S_{(i+1)\bmod M}, \\ &\qquad v \in S_{(j+1)\bmod M}\}, \end{aligned}$$

where S_0 is the set of t vertices added to $Q(n)$.

Lemma 12: $G^3(n)$ is a t-FT graph for $Q(n)$.

Proof: Let F be any subset of $V(G^3(n))$ such that $|F| \leqq t$. Let $F_i = V_i \cap F$ and $t_i = |F_i|$ for any $i \in [M]$, and let $F_M = S_0 \cap F$ and $t_M = |F_M|$. Then, $(F_0, F_1, \ldots, F_M)$ is a partition of F and $|F| = \sum_{i=0}^{M} t_i \leqq t$.

Since $\sum_{j=0}^{i} t_j \leqq t - t_{i+1}$ for any $i \in [M]$, there exists $A_i \subset S_i - F$ such that $|A_i| = \sum_{j=0}^{(i-1)\bmod M} t_j$ for any $i \in [M]$. It follows that $|F_i \cup A_i| = |F_i| + |A_i| = |A_{(i+1)\bmod M}| = \sum_{j=0}^{i} t_j$ for any $i \in [M]^+$, and $|F_0| = |A_1| = t_0$. Thus, there exist bijections

$$\psi_i : \begin{cases} F_0 \rightarrow A_1 & \text{if } i = 0, \\ F_i \cup A_i \rightarrow A_{(i+1)\bmod M} & \text{if } i \in [M]^+. \end{cases}$$

Define the mapping ψ from $V(Q(n))$ to $V(G^3(n) - F)$ as follows:

$$\psi(v) = \begin{cases} v & \text{if } v \notin F \cup A, \\ \psi_0(v) & \text{if } v \in F_0, \\ \psi_i(v) & \text{if } v \in F_i \cup A_i,\ i \in [M]^+, \end{cases}$$

where $A = \bigcup_{i=1}^{k-1} A_i$. It is easy to see that ψ is a one-to-one mapping.

Now we will show that $(\psi(u), \psi(v)) \in E(G^3(n) - F)$ for any $(u,v) \in E(Q(n))$. If $(u,v) \in E(Q(n))$, then $u \in V_i$, $v \in V_j$, $i \in [M]$, and $j \in Neib(i)$. There are three cases as follows.

Case 1 $u, v \notin F \cup A$: Since $\psi(u) = u$ and $\psi(v) = v$, we have $(\psi(u), \psi(v)) \in E(G^3(n) - F)$.

Case 2 $u \notin F \cup A, v \in F \cup A$: Since $\psi(u) = u \in V_i$, $\psi(v) = \psi_j(v) \in A_{(j+1)\bmod M} \subseteq S_{(j+1)\bmod M}$, and $j \in Neib(i)$, we have $(\psi(u), \psi(v)) \in E(G^3(n) - F)$.

Case 3 $u, v \in F \cup A$: Since $\psi(u) = \psi_i(u) \in A_{(i+1)\bmod M} \subseteq S_{(i+1)\bmod M}$, $\psi(v) = \psi_j(u) \in A_{(j+1)\bmod M} \subseteq S_{(j+1)\bmod M}$, and $j \in Neib(i)$, we have $(\psi(u), \psi(v)) \in E(G^3(n) - F)$.

Thus, $(\psi(u), \psi(v)) \in E(G^3(n) - F)$ for any $(u,v) \in E(Q(n))$, and so $G^3(n) - F$ contains $Q(n)$ as a subgraph. Hence $G^3(n)$ is a t-FT graph for $Q(n)$. □

Lemma 13: $\Delta(G^3(n)) = O(N/\log^{c/2} N) + 4ct$.

Proof: Let $deg_3(v)$ denote the degree of $v \in V(G^3(n))$. There are four cases as follows.

Case 1 $v \in V_0$: $deg_3(v) \leqq n + \sum_{j\in Neib(0)} |S_{(j+1)\bmod M}| \leqq n + 2ct$.

Case 2 $v \in S_0$: $deg_3(v) \leqq \sum_{j \in Neib(M-1)} |V_j| + \sum_{j \in Neib(M-1)} |S_{(j+1) \bmod M}| \leqq 2c \max_{j \in [M]} |V_j| + 2ct$.

Case 3 $v \in V_i - S_i$, $i \in [M]^+$: $deg_3(v) \leqq n + \sum_{j \in Neib(i)} |S_{(j+1) \bmod M}| \leqq n + 2ct$.

Case 4 $v \in S_i$, $i \in [M]^+$: $deg_3(v) \leqq n + \sum_{j \in Neib((i-1) \bmod M)} |V_j| + \sum_{j \in Neib(i)} |S_{(j+1) \bmod M}| + \sum_{j \in Neib((i-1) \bmod M)} |S_{(j+1) \bmod M}| \leqq n + 2c \max_{j \in [M]} |V_j| + 4ct$.

Since

$$\max_{j \in [M]} |V_j| = \left(\begin{array}{c} m \\ \lfloor m/2 \rfloor \end{array} \right)^c,$$

we have, by Lemma 4,

$$\max_{j \in [M]} |V_j| = \theta \left(\frac{2^{cm}}{m^{c/2}} \right) = \theta \left(\frac{2^n}{n^{c/2}} \right).$$

Hence $\Delta(G^3(n)) = O(2^n/n^{c/2}) + 4ct$. □

Lemma 14: $\Lambda(G^3(n)) = ct2^{n+1} + ct^2(n/c)^c$

Proof: Since $|S_j| = t$ for any $j \in [M]$, we have

$$\Lambda(G^3(n)) \leqq t \sum_{j \in [M]} |Neib(j)||V_j| + \frac{1}{2} t^2 \sum_{j \in [M]} |Neib(j)|$$
$$\leqq ct2^{n+1} + ct^2 \left(\frac{n}{c} \right)^c.$$

□

By summarizing Lemmas 12, 13, and 14, we have the following theorem.

Theorem 4: Let c be a fixed integer and let $n \geqq 2c$ be a natural number such that $c|n$. Let $t \leqq 2^{c-1}n/c$. Then $G^3(n)$ is a t-FT graph for $Q(n)$ with $2ctN + ct^2(\log N/c)^c$ added edges and maximum degree of $O(N/\log^{c/2} N) + 4ct$. □

We can generalize Theorem 4 for any n. The proof is by a similar argument as the proof of Theorem 4, but is rather complicated and is omitted here.

Theorem 5: Let c be a fixed integer, $n \geqq 2c$ be a natural number, and $r = n \bmod c$. If $t \leqq 2^{c-1}\lfloor n/c \rfloor$, then we can construct a t-FT graph for $Q(n)$ with $2ctN + ct^2\lceil \log N/c \rceil^r \cdot \lfloor \log N/c \rfloor^{c-r}$ added edges and maximum degree of $O(N/\log^{c/2} N) + 4ct$. □

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