Sparse Networks Tolerating Random Faults

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Abstract

This paper proposes a general method to construct a fault-tolerant network G^* for any network G with N processors such that G^* has $O(N)$ processors and contains a fault-free isomorphic copy of G with high probability even if processors fail independently with constant probability. Based on the construction, we also show that we can construct such fault-tolerant networks with $O(N)$ processors and $O(M \log N)$ communication links for a circulant network, hypercube, de Bruijn network, shuffle-exchange network, and cube-connected-cycles with N processors and M communication links.

1. Introduction

This paper considers the following problem in connection with the design of fault-tolerant interconnection networks for multiprocessor systems: Given an N-vertex graph G, construct an $O(N)$-vertex graph G^* with a minimum number of edges such that even after deleting vertices from G^* independently with constant probability, the remaining graph contains G as a subgraph, with probability converging to 1, as $N \to \infty$. G^* is called an RFT (random-fault-tolerant) graph for G. Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G, respectively. Fraigniaud, Kenyon, and Pelc showed that for any N-vertex graph G, there exists an RFT graph for G with $O(|E(G)| \cdot \log^2 N)$ edges, and that there exists a graph G such that any RFT graph for G has $\omega(|E(G)|)$ edges. It is also known that for an N-vertex path[1, 2], cycle[2], and tree with bounded vertex degree[3], there exist RFT graphs with $O(N)$ edges; for an N-vertex mesh and torus[5], there exist RFT graphs with $O(N \log\log N)$ edges; and for an N-vertex tree, there exists an RFT graph with $O(N \log N)$ edges [2].

In this paper, we propose a general method to construct an RFT graph for any graph. Based on the construction, we show that if G is an N-vertex circulant graph, hypercube, de Bruijn graph, shuffle-exchange graph, or cube-connected-cycles, we can construct an RFT graph for G with $O(|E(G)| \cdot \log N)$ edges.

2. General Construction

For any positive integer k, let $[k] = \{0, 1, \ldots, k-1\}$. For any set of S, a collection $\mathcal{S} = \{S_0, S_1, \ldots, S_{k-1}\}$ of subsets of S is a partition of S if $\bigcup_{i \in [k]} S_i = S$ and $S_i \cap S_j = \emptyset$ for any $i \neq j$.

Let G be any N-vertex graph. For any partition $\mathcal{V} = \{V_0, V_1, \ldots, V_{k-1}\}$ of $V(G)$, define

$$\Lambda(G, \mathcal{V}) = \{(i,j) |\ \exists (u,v) \in E(G) (u \in V_i, v \in V_j)\}$$

and

$$\lambda(G, \mathcal{V}) = |\Lambda(G, \mathcal{V})|.$$

Let $0 < p < 1$ be the probability for each vertex to be deleted. The deleted and undeleted vertices are said to be faulty and fault-free, respectively.

Let $\mathcal{V} = \{V_0, V_1, \ldots, V_{k-1}\}$ be any partition of $V(G)$ such that $|V_i| \leq \alpha \ln N$ for any $i \in [k]$ and $k \leq \beta N / \ln N$ for some fixed positive numbers α and β. Let $V_0^*, V_1^*, \ldots,$ and V_{k-1}^* be k sets such that $|V_i^*| = \lceil \gamma \ln N \rceil$ for any $i \in [k]$ and $V_i^* \cap V_j^* = \emptyset$ for any $i \neq j$, where

$$\gamma = \frac{(\sqrt{2\alpha + 1} + 1)^2}{2(1-p)}.$$

Note that γ is fixed since α and p are fixed. Then, $G^*[\mathcal{V}]$ is the graph defined as follows:

$$\begin{aligned} V(G^*[\mathcal{V}]) &= V_0^* \cup V_1^* \cup \cdots \cup V_{k-1}^*; \\ E(G^*[\mathcal{V}]) &= \left\{ (u^*, v^*) \;\middle|\; \begin{array}{l} u^* \in V_i^*,\ v^* \in V_j^*, \\ (i,j) \in \Lambda(G, \mathcal{V}) \end{array} \right\}. \end{aligned}$$

Theorem 1 *Let G be any N-vertex graph, and let $\mathcal{V} = \{V_0, V_1, \ldots, V_{k-1}\}$ be any partition of $V(G)$ such that $|V_i| = O(\ln N)$ and $k = O(N/\ln N)$. Then $G^*[\mathcal{V}]$ is an RFT graph for G with $O(\lambda(G, \mathcal{V}) \cdot \log^2 N)$ edges.*

Proof : We prove the theorem by a series of lemmas. It is easy to see the following two lemmas.

Lemma 1 $|V(G^*[\mathcal{V}])| \leq \dfrac{\beta N}{\ln N} \cdot \lceil \gamma \ln N \rceil$.

Lemma 2 $|E(G^*[\mathcal{V}])| \leq \lambda(G, \mathcal{V}) \cdot \lceil \gamma \ln N \rceil^2$.

Now we prove that $G^*[\mathcal{V}]$ is an RFT graph for G. We need a few probabilistic notations and lemmas.

For any event E, let $\mathrm{Prob}[E]$ denote the probability of E. For any random variable X and real number r, let $\{X \leq r\}$ denote the event that $X \leq r$. The probability of $\{X \leq r\}$ is denoted by $\mathrm{Prob}[X \leq r]$ instead of $\mathrm{Prob}[\{X \leq r\}]$. The following inequality is well-known as Chernoff Bound.

Lemma 3 [4] *Let X be the binomial variable with parameters m and q, that is, the number of successes in m Bernoulli trials with probabilities q for success and $1 - q$ for failure. Then, for any constant $0 < \epsilon < 1$,*

$$\mathrm{Prob}[X \leq (1-\epsilon)qm] \leq \exp(-\frac{1}{2}\epsilon^2 qm).$$

Lemma 4 *Let Y_i be the number of fault-free vertices of V_i^*. Then, for any $i \in [k]$,*

$$\mathrm{Prob}[Y_i \leq \alpha \ln N] \leq \frac{1}{N}.$$

Moreover,

$$\mathrm{Prob}[\bigcup_{i=0}^{k-1} \{Y_i \leq \alpha \ln N\}] \leq \frac{\beta}{\ln N}.$$

Proof : Set $\epsilon = 2/(\sqrt{2\alpha+1}+1)$, $q = 1-p$, and $m = \lceil \gamma \ln N \rceil$. Since $0 < \epsilon < 1$,

$$\begin{aligned}
&(1-\epsilon)qm \\
&\geq \frac{\sqrt{2\alpha+1}-1}{\sqrt{2\alpha+1}+1} \cdot (1-p) \cdot \frac{(\sqrt{2\alpha+1}+1)^2}{2(1-p)} \cdot \ln N \\
&= \frac{2\alpha}{(\sqrt{2\alpha+1}+1)^2} \cdot \frac{(\sqrt{2\alpha+1}+1)^2}{2} \cdot \ln N \\
&= \alpha \ln N,
\end{aligned}$$

and

$$\begin{aligned}
&\frac{1}{2}\epsilon^2 qm \\
&\geq \frac{1}{2} \cdot \frac{4}{(\sqrt{2\alpha+1}+1)^2} \cdot (1-p) \cdot \frac{(\sqrt{2\alpha+1}+1)^2}{2(1-p)} \cdot \ln N \\
&= \ln N,
\end{aligned}$$

we obtain by Lemma 3 that

$$\begin{aligned}
\mathrm{Prob}[Y_i \leq \alpha \ln N] &\leq \mathrm{Prob}[Y_i \leq (1-\epsilon)qm] \\
&\leq \exp(-\frac{1}{2}\epsilon^2 qm) \\
&\leq \frac{1}{N}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\mathrm{Prob}[\bigcup_{i=0}^{k-1} \{Y_i \leq \alpha \ln N\}] &\leq \sum_{i=0}^{k-1} \mathrm{Prob}[Y_i \leq \alpha \ln N] \\
&\leq \frac{\beta}{\ln N}.
\end{aligned}$$

∎

Lemma 5 *$G^*[\mathcal{V}]$ is an RFT graphs for G.*

Proof : Let ϕ be a one-to-one mapping from $V(G)$ to $V(G^*[\mathcal{V}])$ such that $\phi(v)$ is a fault-free vertex of V_i^* for any $v \in V_i$. By Lemma 4, such ϕ exists with probability at least $1 - (\beta/\ln N)$.

Now we show that $(\phi(u), \phi(v)) \in E(G^*[\mathcal{V}])$ for any $(u, v) \in E(G)$. Let $u \in V_i$ and $v \in V_j$. Then, $(i, j) \in \Lambda(G, \mathcal{V})$. Since $\phi(u) \in V_i^*$ and $\phi(v) \in V_j^*$, we conclude that $(\phi(u), \phi(v)) \in E(G^*[\mathcal{V}])$. Hence $G^*[\mathcal{V}]$ is an RFT graphs for G. ∎

This completes the proof of Theorem 1. ∎

Since $\lambda(G, \mathcal{V}) \leq |E(G)|$, we obtain the following corollary.

Corollary 1 *Let G be any N-vertex graph, and let $\mathcal{V} = \{V_0, V_1, \ldots, V_{k-1}\}$ be any partition of $V(G)$ such that $|V_i| = O(\ln N)$ and $k = O(N/\ln N)$. $G^*[\mathcal{V}]$ is an RFT graph for G with $O(|E(G)| \cdot \log^2 N)$ edges.*

This corollary means that, for any N-vertex graph G, there exists an RFT graph for G with $O(|E(G)| \cdot \log^2 N)$ edges, which is also obtained in [2].

3. RFT Graphs for Circulant Graphs

Let N be a positive integer and let $S \subset [N]$. The N-vertex circulant graph with connection set S, denoted by $C_N(S)$, is the graph defined as follows:

$$\begin{aligned}
V(C_N(S)) &= [N]; \\
E(C_N(S)) &= \{(u, v) |\ \exists s \in S (v = (u \pm s) \bmod N)\}.
\end{aligned}$$

An edge (u, v) is said to be of offset s if $v = (u \pm s) \bmod N$.

It is easy to see the following lemma.

Lemma 6 *Let $S' = \{s \mid s \in S$ and $s \leq N/2\} \cup \{N - s \mid s \in S$ and $s > N/2\}$. Then $C_N(S')$ is isomorphic to $C_N(S)$. Moreover,*

$$|E(C_N(S))| = \begin{cases} (2|S'|-1)N/2 & \text{if } N/2 \in S, \\ |S'|N & \text{otherwise.} \end{cases}$$

Let $c_N = \lceil \log N \rceil$ and $k_N = \lceil N/c_N \rceil$. Define $U_i = \{v \in [N] \mid \lfloor v/c_N \rfloor = i\}$ for any $i \in [k_N]$. Then, $\mathcal{U}_N = \{U_0, U_1, \ldots, U_{k_N-1}\}$ is a partition of $[N]$ such that $|U_i| \leq c_N \leq \log N + 1$ for any $i \in [k_N]$ and $|\mathcal{U}_N| = k_N = \lceil N/c_N \rceil \leq (N/\log N) + 1$.

Theorem 2 *$C_N^*(S)[\mathcal{U}_N]$ is an RFT graph for $C_N(S)$ with $O(|E(C_N(S))| \cdot \log N)$ edges.*

Proof : By Lemma 6, we may assume that if $s \in S$ then $s \leq N/2$. Thus, By Theorem 1 and Lemma 6, it suffices to prove that $\lambda(C_N(S), \mathcal{U}_N) = O(|S|N/\log N)$.

Consider any edge $(u, v) \in E(C_N(S))$ of offset $s \in S$. Assume without loss of generality that $v = (u + s) \bmod N$. Let $u = ic_N + a$ and $v = jc_N + b$, where $0 \leq a, b < c_N$. Then $u \in U_i$ and $v \in U_j$. We have the following two cases:

(i) $u < N - s$: We have $v = u + s$. Then, $jc_N + b = ic_N + a + s$, and so $j = i + (s + a - b)/c_N$. Since i and j are integers, $l = (s + a - b)/c_N$ is an integer. Thus, $\lfloor s/c_N \rfloor = l + \lfloor (b - a)/c_N \rfloor$. that is $l = \lfloor s/c_N \rfloor - \lfloor (b-a)/c_N \rfloor$. Since $0 \leq a, b < c_N$, $\lfloor (b-a)/c_N \rfloor = 0$ or -1, and so $l = \lfloor s/c_N \rfloor$ or $l = \lfloor s/c_N \rfloor + 1$. Hence $j = i + \lfloor s/c_N \rfloor$ or $j = i + \lfloor s/c_N \rfloor + 1$.

(ii) $u \geq N - s$: We have $v = u + s - N$. Then, $jc_N + b = ic_N + a + s - N$, and so $j = i + (a - b + s - N)/c_N$. Since $N = k_N c_N - d$ by the definition of k_N, we obtain $j = i - k_N + (a - b + d + s)/c_N$, where $0 \leq d < c_N$. Since i and j are integers, $l = (a - b + d + s)/c_N$ is an integer. Thus, $\lfloor s/c_N \rfloor = l + \lfloor (b - a - d)/c_N \rfloor$, that is $l = \lfloor s/c_N \rfloor - \lfloor (b - a - d)/c_N \rfloor$. Since $0 \leq a, b, d < c_N$, $\lfloor (b - a - d)/c_N \rfloor = 0, -1$, or -2, and so we have $l = \lfloor s/c_N \rfloor$, $l = \lfloor s/c_N \rfloor + 1$, or $l = \lfloor s/c_N \rfloor + 2$. Hence $j = (i + \lfloor s/c_N \rfloor) \bmod k_N$, $j = (i + \lfloor s/c_N \rfloor + 1) \bmod k_N$, or $j = (i + \lfloor s/c_N \rfloor + 2) \bmod k_N$.

Thus,

$$\Lambda(C_N(S), \mathcal{U}_N) \subseteq \left\{ (i,j) \;\middle|\; \begin{array}{l} i, j \in [k_N],\ s \in [S],\ r \in [3], \\ j = (i \pm \lfloor s/c_N \rfloor + r) \bmod k_N \end{array} \right\}$$

and we have

$$\lambda(C_N(S), \mathcal{U}_N) \leq 3|S|k_N \leq \frac{3|S|N}{\log N} + 3|S| = O(\frac{|S|N}{\log N}).$$

$\blacksquare$

4. RFT Graphs for Hypercubic Graphs

For any $v = [v_n, v_{n-1}, \ldots, v_1] \in [2]^n$, let

$$\begin{aligned} \sigma(v) &= [v_{n-1}, \ldots, v_1, v_n], \\ \chi_i(v) &= [v_n, \ldots, v_{i+1}, \overline{v_i}, v_{i-1}, \ldots, v_1], \text{ and} \\ \rho_i(v) &= [v_i, \ldots, v_1], \end{aligned}$$

where $\overline{v_i}$ denotes the complement of v_i, that is $\overline{v_i} = 1$ if $v_i = 0$, and $\overline{v_i} = 0$ otherwise.

Let

$$V_x = \{v \in [2]^n \mid \rho_{n - \lceil \log n \rceil}(v) = x\}$$

for any $x \in [2]^{n - \lceil \log n \rceil}$ and let

$$\mathcal{V}_n = \{V_x \mid x \in [2]^{n - \lceil \log n \rceil}\}.$$

Then $\mathcal{V}_n$ is a partition of $[2]^n$ such that $|V_x| \leq 2 \log N$ for any $x \in [2]^{n - \lceil \log n \rceil}$ and $|\mathcal{V}_n| \leq N/\log N$, where $N = |[2]^n| = 2^n$.

4.1. RFT Graphs for Hypercubes

The n-cube (n-dimensional cube) $Q(n)$ is the graph defined as follows:

$$\begin{aligned} V(Q(n)) &= [2]^n; \\ E(Q(n)) &= \{(u, v) \mid v = \chi_i(u),\ 1 \leq i \leq n\}. \end{aligned}$$

It is easy to see that $|V(Q(n))| = N$ and $|E(Q(n))| = (N \log N)/2$, where $N = 2^n$. An edge (u, v) is called an i-edge if $v = \chi_i(u)$. A graph G is called a hypercube if G is isomorphic to $Q(n)$ for some n.

Theorem 3 *$Q^*(n)[\mathcal{V}_n]$ is an RFT graph for $Q(n)$ with $O(N \log^2 N)$ edges.*

Proof : By Theorem 1, it suffices to prove that $\lambda(Q(n), \mathcal{V}_n) = O(N)$.

Consider any i-edge $(u, v) \in E(Q(n))$. Let $x = \rho_{n - \lceil \log n \rceil}(u)$ and $y = \rho_{n - \lceil \log n \rceil}(v)$. It is easy to see that $y = \chi_i(x)$ if $1 \leq i \leq n - \lceil \log n \rceil$, and $x = y$ otherwise. Thus,

$$\Lambda(Q(n), \mathcal{V}_n) \subseteq \{(x, y) \mid y = \chi_i(x) \text{ or } x = y\},$$

and we have

$$\begin{aligned} \lambda(Q(n), \mathcal{V}_n) &\leq \{\frac{1}{2}(n - \lceil \log n \rceil) + 1\} \cdot 2^{n - \lceil \log n \rceil} \\ &\leq \{\frac{1}{2}(\log N - \log \log N) + 1\} \cdot \frac{N}{\log N} \\ &= O(N). \end{aligned}$$

$\blacksquare$

4.2. RFT Graphs for de Bruijn Graphs

The n-dimensional de Bruijn graph $dB(n)$ is the graph defined as follows:

$$\begin{aligned} V(dB(n)) &= [2]^n; \\ E(dB(n)) &= \{(u,v) \mid v = \sigma(u) \text{ or } u = \sigma(v)\} \\ & \cup \left\{ (u,v) \;\middle|\; \begin{array}{c} v = \chi_1(\sigma(u)) \\ \text{or} \\ u = \chi_1(\sigma(v)) \end{array} \right\}. \end{aligned}$$

It is easy to see that $|V(dB(n))| = N$ and $|E(dB(n))| = 2N$, where $N = 2^n$.

Theorem 4 *$dB^*(n)[\mathcal{V}_n]$ is an RFT graph for $dB(n)$ with $O(N \log N)$ edges.*

Proof : By Theorem 1, it suffices to prove that $\lambda(dB(n), \mathcal{V}_n) = O(N/\log N)$.

Consider any edge $(u,v) \in E(dB(n))$. Assume without loss of generality that $v = \sigma(u)$ or $v = \chi_1(\sigma(u))$. Let $x = \rho_{n-\lceil \log n \rceil}(u)$ and $y = \rho_{n-\lceil \log n \rceil}(v)$. It is easy to see that $y = \sigma(x)$ or $y = \chi_1(\sigma(x))$. Thus,

$$\begin{aligned} \Lambda(dB(n), \mathcal{V}_n) &\subseteq \{(x,y) \mid y = \sigma(x) \text{ or } x = \sigma(y)\} \\ & \cup \left\{ (x,y) \;\middle|\; \begin{array}{c} y = \chi_1(\sigma(x)) \\ \text{or} \\ x = \chi_1(\sigma(y)) \end{array} \right\}, \end{aligned}$$

and we have

$$\lambda(dB(n), \mathcal{V}_n) \le 2^{n-\lceil \log n \rceil + 1} \le \frac{2N}{\log N} = O(\frac{N}{\log N}).$$

∎

4.3. RFT Graphs for Shuffle-Exchange Graphs

The n-dimensional shuffle-exchange graph $SE(n)$ is the graph defined as follows:

$$\begin{aligned} V(SE(n)) &= [2]^n; \\ E(SE(n)) &= \{(u,v) \mid v = \sigma(u) \text{ or } u = \sigma(v)\} \\ & \cup \{(u,v) \mid v = \chi_1(u)\}. \end{aligned}$$

It is easy to see that $|V(SE(n))| = N$ and $|E(SE(n))| = 3N/2$, where $N = 2^n$.

Theorem 5 *$SE^*(n)[\mathcal{V}_n]$ is an RFT graph for $SE(n)$ with $O(N \log N)$ edges.*

Proof : By Theorem 1, it suffices to prove that $\lambda(SE(n), \mathcal{V}_n) = O(N/\log N)$.

Consider any edge $(u,v) \in E(SE(n))$. Let $x = \rho_{n-\lceil \log n \rceil}(u)$ and $y = \rho_{n-\lceil \log n \rceil}(v)$. If $v = \sigma(u)$ then we have $y = \sigma(x)$ or $y = \chi_1(\sigma(x))$. If $v = \chi_1(u)$ then $x = y$. Thus,

$$\begin{aligned} \Lambda(SE(n), \mathcal{V}_n) &\subseteq \{(x,y) \mid y = \sigma(x) \text{ or } x = \sigma(y)\} \\ & \cup \left\{ (x,y) \;\middle|\; \begin{array}{c} y = \chi_1(\sigma(x)) \\ \text{or} \\ x = \chi_1(\sigma(y)) \end{array} \right\} \\ & \cup \{(x,y) \mid x = y\}, \end{aligned}$$

and we have

$$\lambda(SE(n), \mathcal{V}_n) \le 3 \cdot 2^{n-\lceil \log n \rceil} \le \frac{3N}{\log N} = O(\frac{N}{\log N}).$$

∎

4.4. RFT Graphs for CCC's

The n-dimensional cube-connected-cycles(CCC), denoted by $CCC(n)$, is the graph defined as follows:

$$\begin{aligned} V(CCC(n)) &= [2]^n \times [n]; \\ E(CCC(n)) &= \{([v,i],[v,j]) \mid j = (i \pm 1) \bmod n\} \\ & \cup \{([u,i],[v,i]) \mid v = \chi_{i+1}(u)\}, \end{aligned}$$

where $u, v \in [2]^n$ and $i, j \in [n]$. It is easy to see that $|V(CCC(n))| = N$ and $|E(CCC(n))| = 3N/2$, where $N = n2^n$.

Let

$$V'_{[x,i]} = \{[u,i] \in V(CCC(n)) \mid \rho_{n-\lceil \log n \rceil}(u) = x\}$$

for any $x \in [2]^{n-\lceil \log n \rceil}$ and $i \in [n]$ and let

$$\mathcal{V}'_n = \{V_{[x,i]} \mid x \in [2]^{n-\lceil \log n \rceil},\ i \in [n]\}.$$

It is easy to see that $\mathcal{V}'_n$ is a partition of $V(CCC(n))$ such that $|V_{[x,i]}| \le 2 \log N$ for any $x \in [2]^{n-\lceil \log n \rceil}$ and $i \in [n]$, and $|\mathcal{V}'_n| \le 2N/\log N$.

Theorem 6 *$CCC^*(n)[\mathcal{V}'_n]$ is an RFT graph for $CCC(n)$ with $O(N \log N)$ edges.*

Proof : By Theorem 1, it suffices to prove that $\lambda(CCC(n), \mathcal{V}_n) = O(N/\log N)$.

Consider any edge $([u,i],[v,j]) \in E(CCC(n))$. Let $x = \rho_{n-\lceil \log n \rceil}(u)$ and $y = \rho_{n-\lceil \log n \rceil}(v)$. If $u = v$ then $x = y$. If $v = \chi_{i+1}(u)$ then $y = \chi_{i+1}(x)$ or $x = y$. Thus,

$$\begin{aligned} & \Lambda(CCC(n), \mathcal{V}'_n) \\ & \subseteq \{([x,i],[y,j] \mid x = y, j = (i \pm 1) \bmod n\} \\ & \quad \cup \{[x,i],[y,j] \mid y = \chi_{i+1}(x), i = j\} \\ & \quad \cup \{[x,i],[y,j] \mid x = y, i = j\}, \end{aligned}$$

and we have

$$\begin{aligned}\lambda(CCC(n), \mathcal{V}'_n) &\leq \frac{5}{2} n 2^{n-\lceil \log n \rceil} \\ &\leq \frac{5N}{\log N} \\ &= O(\frac{N}{\log N}).\end{aligned}$$

∎

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