# On-Line Multicasting in All-Optical Networks

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**Abstract.** We consider the routing for a special type of communication requests, called a multicast, consisting of a fixed source and a multiset of destinations in a wavelength division multiplexing all optical network. We prove a min-max equality that the minimum number of wavelengths necessary for routing a multicast is equal to the maximum of the average number of paths that share a link in a cut of the network. Based on the min-max equality above, we propose an on-line algorithm for routing a multicast, and show that the competitive ratio of our algorithm is equal to the ratio of the degree of the source to the link connectivity of the network. We also show that 4/3 is a lower bound for the competitive ratio of an on-line algorithm for routing a multicast.

## 1 Introduction

A WDM (Wavalength Division Multiplexing) all-optical network consists of routing nodes interconnected by point-to-point unidirectional fiber-optic links, which support a certain number of wavelengths. The same wavelength on two input ports cannot be routed to a same output port due to the interference. A fundamental problem for WDM all-optical networks is the optical routing, which assigns a path and a wavelength for each communication request in such a way that no two paths that traverse a common link are assigned the same wavelength by using as few wavelengths as possible. This paper considers the on-line optical routing for a special collection of communication requests called a multicast.

A WDM all-optical network is modeled as a symmetric digraph (directed graph) $G$ with vertex set $V(G)$ and arc (directed edge) set $A(G)$ such that if $(u, v) \in A(G)$ then $(v, u) \in A(G)$, where the vertices represent the routing nodes and each arc represents a point-to-point unidirectional fiber-optic link connecting a pair of routing nodes.

Let $P(x, y)$ denote a dipath (directed path) in $G$ from the vertex $x$ to $y$ which consistis of consecutive arcs beginning at $x$ and ending at $y$. A request is an ordered pair of vertices $(x, y)$ in $G$ corresponding to a message to be sent from $x$ to $y$, and an instance $I$ is a collection (multiset) of requests. A routing for an instance $I$ is a collection of dipaths $R = \{P(x, y)|(x, y) \in I\}$.

Given a symmetric digraph $G$, an instance $I$, and a routing $R$ for $I$, $\omega(G, I, R)$ is the minimum number of wavelengths that can be assigned to the dipaths in $R$,

P. Eades and T. Takaoka (Eds.): ISAAC 2001, LNCS 2223, pp. 99–106, 2001.
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so that no two dipaths sharing an arc have the same wavelength. Let $\omega(G,I)$ denote the smallest $\omega(G,I,R)$ over all routings $R$ for $I$. The load of an arc $\alpha \in A(G)$ in $R$, denoted by $\pi(G,I,R,\alpha)$, is the number of dipaths in $R$ containing $\alpha$. Let $\pi(G,I,R)$ denote the largest $\pi(G,I,R,\alpha)$ over all arcs $\alpha \in A(G)$, and $\pi(G,I)$ denote the smallest $\pi(G,I,R)$ over all routings $R$ for $I$. It is known that computing $\omega(G,I)$ and $\pi(G,I)$ is NP-hard in general [2]. It is not difficult to see that $\omega(G,I) \geq \pi(G,I)$ for an instance $I$ in a symmetric digraph $G$ and that the inequality can be strict in general [2].

Beauquier, Hell, and Perennes [3] proved that for a multicast $I$ in a symmetric digraph $G$, $\omega(G,I) = \pi(G,I)$ and both $\omega(G,I)$ and $\pi(G,I)$ can be computed in polynominal time. An instance $I$ is called a multicast if $I$ is of the form $\{(x,y)|y \in Y\}$ for a fixed vertex $x \in V(G)$, called the source, and a collection $Y$ of vertices in $V(G)$, called the destinations.

This paper shows a min-max equality on $\omega(G,I)$ for a multicast $I$ in a symmetric digraph $G$ by means of the cut in $G$. For a digraph $G$ and a nonempty proper subset $S \subset V(G)$, a cut $(S,\overline{S})$ is the set of arcs beginning in $S$ and ending in $\overline{S}$, where $\overline{S} = V(G) - S$. For a multicast $I = \{(x,y)|y \in Y\}$ and a cut $(X,\overline{X})$ with $x \in X \subset V(G)$, let $\mu(G,I,X)$ denote $\lceil |Y \cap \overline{X}|/|(X,\overline{X})| \rceil$, and $\mu(G,I)$ denote the largest $\mu(G,I,X)$ over all cuts $(X,\overline{X})$ with $x \in X \subset V(G)$. Notice that $\mu(G,I,X)$ is a lower bound on the average load of an arc in $(X,\overline{X})$ for any routing for $I$. We prove a min-max equality that $\omega(G,I) = \mu(G,I)$, which is used as a basis for on-line multicasting. Let $\delta(x)$ denote the outdegree of $x$ and $\lambda(x)$ denote $\min\{|(X,\overline{X})||x \in X \subset V(G)\}$. Notice that $\delta(x) \geq \lambda(x)$. If $I$ is a broadcast, that is $I = \{(x,y)|y \in V(G) - x\}$ and $\delta(x) = \lambda(x)$ then our min-max equality implies that $\omega(G,I) = \lceil |V(G) - 1|/\delta(x) \rceil$, which is essentially Theorem 3.1 in [4] proved by Bermond, Gargano, Perennes, Rescigno, and Vaccaro.

Given a symmetric digraph $G$ and a sequence of requests $(x_i, y_i)$, an on-line algorithm assigns a dipath $P(x_i, y_i)$ and a wavelength to $P(x_i, y_i)$, so that no two dipaths sharing an arc are assigned the same wavelength. The performance measure for an on-line algorithm is the competitive ratio defined as the worst-case ratio over all request sequences between the number of wavelengths used by the on-line algorithm and the optimal number of wavelengths necessary on the same sequence. Bartal and Leonardi [1] showed on-line algorithms with competitive ratio of $O(\log N)$ for any instances in $N$-vertex digraphs associated with meshs, trees, and trees of rings, where the digraph associated with a graph $H$ is the symmetric digraph obtained when each edge $e$ of $H$ is replaced by two oppositely oriented arcs with the same ends as $e$. They also proved a matching lower bound of $\Omega(\log N)$ for digraphs associated with meshes, and a lower bound of $\Omega(\log N/\log\log N)$ for digraphs associated with trees and trees of rings [1].

We show here an on-line algorithm for a multicast $I = \{(x,y)|y \in Y\}$ in a symmetric digraph $G$. We prove that the competitive ratio of our algorithm is $\lceil \delta(x)/\lambda(x) \rceil$. It follows that if $\delta(x) = O(1)$ then the competitive ratio of our algorithm is $O(1)$. Moreover, if $\delta(x) = \lambda(x)$ then our algorithm is optimal. We also show a complementary result that if $\delta(x) > \lambda(x)$ then there is no optimal on-line algorithm. Moreover, we show that the competitive ratio of any on-line algorithm is at least 4/3. We also consider the dynamic multicasting.

## 2 Off-Line Multicasting

We prove in this section the following min-max equality, which will be used in the subsequent sections.

**Theorem 1.** *$\omega(G, I) = \mu(G, I)$ for a multicast $I$ in a symmetric digraph $G$.*

### 2.1 Proof of Theorem 1

Let $G$ be a symmetric digraph and $I = \{(x, y)|y \in Y\}$ be a multicast in $G$.

**Proof of $\omega(G, I) \geq \mu(G, I)$.** It is well-known and easily verified that

$$\omega(G, I) \geq \pi(G, I). \tag{1}$$

Since $\mu(G, I, X)$ is a lower bound on the average load of an arc in a cut $(X, \overline{X})$ with $x \in X \subset V(G)$ for any routing $R$ for $I$, we have

$$\pi(G, I, R) \geq \mu(G, I, X)$$

for any routing $R$ for $I$ and any cut $(X, \overline{X})$ with $x \in X \subset V(G)$. Thus, it follows that

$$\pi(G, I) \geq \mu(G, I). \tag{2}$$

Combining (1) and (2), we have

$$\omega(G, I) \geq \mu(G, I).$$

**Proof of $\omega(G, I) \leq \mu(G, I)$.** It is proved in [3] that for a multicast $I = \{(x, y)|y \in Y\}$ in a symmetric digraph $G$ we have

$$\omega(G, I) = \pi(G, I), \tag{3}$$

by using flow networks derived from $G$.

In a flow network, we denote by $c(u, v)$ the capacity of an arc $(u, v)$, and by $c(T, \overline{T})$ the capacity of a cut $(T, \overline{T})$. Although $Y$ is a collection (multiset) in general, we assume without loss of generality that $Y$ is just a set, as mentioned in [3].

In order to compute $\pi(G, I)$ the following flow network $F_p$ is introduced in [3]. Let $s$ and $t$ be two new vertices which will be the source and sink in $F_p$, respectively. The flow network $F_p$ is defined as follows:

$$
\begin{aligned}
&V(F_p) = \{s,t\} \cup V(G)\\
&A(F_p) = \{(s,x)\} \cup A(G) \cup (\bigcup_{y\in Y}\{(y,t)\})\\
&c(s,x) = \infty\\
&c(u,v) = p \text{ for all } (u,v) \in A(G)\\
&c(y,t) = 1 \text{ for all } y \in Y.
\end{aligned}
$$

The following theorem is immediate from the definitions.

**Theorem I** [3] *$\pi(G,I) \le p$ if and only if $F_p$ has a flow of value $|Y|$.*

By (3) and Theorem I above, it suffices to show that $F_{\mu(G,I)}$ has a flow of value $|Y|$. We prove this by showing that any cut in $F_{\mu(G,I)}$ separating $s$ and $t$ has capacity at least $|Y|$. Any cut in $F_{\mu(G,I)}$ separating $s$ and $t$ can be represented as $(S\cup\{s\}, \overline{S}\cup\{t\})$ for a subset $S$ of $V(G)$ and $\overline{S} = V(G) - S$. It is easy to see that

$$
c(S\cup\{s\}, \overline{S}\cup\{t\}) = \begin{cases} |Y\cap S| + \mu(G,I)\cdot|(S,\overline{S})| & \text{if } x\in S\\ \infty & \text{if } x \in \overline{S}\end{cases}
$$

where $(S,\overline{S})$ is a cut in $G$. It follows that we may assume that $x \in S$. Then we have

$$
\begin{aligned}
c(S\cup\{s\}, \overline{S}\cup\{t\}) &= |Y\cap S| + \mu(G,I)\cdot|(S,\overline{S})|\\
&= |Y\cap S| + \max\left\{\left\lceil\frac{|Y\cap\overline{X}|}{|(X,\overline{X})|}\right\rceil \;\middle|\; x\in X\subset V(G)\right\}\cdot|(S,\overline{S})|\\
&\ge |Y\cap S| + \left\lceil\frac{|Y\cap\overline{S}|}{|(S,\overline{S})|}\right\rceil\cdot|(S,\overline{S})|\\
&\ge |Y\cap S| + \frac{|Y\cap\overline{S}|}{|(S,\overline{S})|}\cdot|(S,\overline{S})|\\
&= |Y\cap S| + |Y\cap\overline{S}| = |Y|,
\end{aligned}
$$

as desired.

## 3 On-Line Multicasting

### 3.1 Upper Bounds

Let $G$ be a symmetric digraph, and $(x,y_1),(x,y_2),\cdots,(x,y_j),\cdots$ be a sequence of multicast requests in $G$. Let $I_j$ denote the collection $\{(x,y_1),(x,y_2),\cdots,(x,y_j)\}$, and $Y_j$ denote the collection $\{y_1,y_2,\cdots,y_j\}$. We assume without loss of generality that $x$ is not a cut-vertex in $G$. We also assume that the wavelengths are labeled with positive integers. Our on-line algorithm is based on the following classic theorem due to Edmonds [5]. For a vertex $u$ of a digraph $G$, $u$-arborescence $H(u)$ in $G$ is an acyclic spanning subdigraph of $G$ such that for every vertex $v\in V(G)$ there is exactly one dipath in $H(u)$ from $u$ to $v$.

**Theorem II** [5] *For a digraph $G$ and a vertex $u \in V(G)$, the maximum number of arc-disjoint $u$-arborescences in $G$ is equal to $\lambda(u)$.*

Let $\mathcal{H} = \{H_1(x), H_2(x), \cdots, H_{\lambda(x)}(x)\}$ be a set of arc-disjoint $x$-arborescences in $G$. For each request, our on-line algorithm, called ARB, assigns a dipath in an $x$-arborescence in $\mathcal{H}$. Given a request $(x, y_j)$, ARB finds an $x$-arborescence $H_k(x)$ such that the number of dipaths in $H_k(x)$ assigned to the existing requests is minimal, assigns the unique dipath $P(x, y_j)$ in $H_k(x)$, and assigns the lowest available wavelength to $P(x, y_j)$.

**Theorem 2.** *The competitive ratio of* ARB *is* $\lceil \delta(x)/\lambda(x) \rceil$.

*Proof.* From Theorem 1, we have that for any $j$,

$$\begin{aligned}
\omega(G, I_j) &= \mu(G, I_j) \\
&= \max \left\{ \left\lceil \frac{|Y_j \cap \overline{X}|}{|(X, \overline{X})|} \right\rceil \,\middle|\, x \in X \subset V(G) \right\} \\
&\geq \left\lceil \frac{|Y_j \cap (V(G) - \{x\})|}{|(\{x\}, V(G) - \{x\})|} \right\rceil \\
&= \left\lceil \frac{|Y_j|}{\delta(x)} \right\rceil \\
&\geq \frac{|Y_j|}{\delta(x)}.
\end{aligned}$$

Let $\omega(G, I_j, \mathrm{ALG})$ denote the number of wavelengths used by an on-line algorithm ALG for $I_j$. We have that

$$\begin{aligned}
\omega(G, I_j, \mathrm{ARB}) &= \left\lceil \frac{|Y_j|}{\lambda(x)} \right\rceil \\
&\leq \left\lceil \frac{\omega(G, I_j) \cdot \delta(x)}{\lambda(x)} \right\rceil \\
&\leq \left\lceil \frac{\delta(x)}{\lambda(x)} \right\rceil \cdot \omega(G, I_j),
\end{aligned}$$

as desired. □

The following corollaries are immediate. An on-line algorithm ALG is said to be optimal for $G$ if $\omega(G, I_j, \mathrm{ALG}) = \omega(G, I_j)$ for any $j$.

**Corollary 1.** *If $\delta(x)$ is $O(1)$ then the competitive ratio of* ARB *is $O(1)$.*

**Corollary 2.** *If $\delta(x) = \lambda(x)$ then* ARB *is optimal for $G$.*

**Corollary 3.** ARB *is optimal for digraphs associated with trees, cycles, tori, hypercubes, and cube-connected cycles.*

### 3.2 Lower Bounds

The following is a complementary result to Corollary 2.

**Theorem 3.** *If $\delta(x) > \lambda(x)$ then there is no on-line algorithm optimal for $G$.*

*Proof.* We prove the theorem by contradiction. Let $G$ be a symmetric digraph, and $x$ be a vertex in $G$ with $\delta(x) > \lambda(x)$. Assume that there is an on-line algorithm ALG optimal for $G$. Let $(X, \overline{X})$ be a cut in $G$ such that $x \in X \subset V(G)$ and $|(X, \overline{X})| = \lambda(x)$, and $v$ be a vertex in $\overline{X}$. We denote the arcs with tail $x$ by $(x, u_1), (x, u_2), \cdots, (x, u_{\delta(x)})$. We consider the following sequence of requests:

$$(x, u_1), (x, u_2), \cdots, (x, u_{\delta(x)}), \underbrace{(x, v), (x, v), \cdots, (x, v)}_{\lambda(x)+1}.$$

Since ALG is optimal for $G$, ALG assigns for the requests $(x, u_i)$ arc-disjoint dipaths $P(x, u_i)$ and the same wavelength, say $w$, to the dipaths $P(x, u_i)$ $(1 \le i \le \delta(x))$. Notice that each arc $(x, u_i)$ is contained in the dipaths assigned wavelength $w$ $(1 \le i \le \delta(x))$. Since $|(X, \overline{X})| = \lambda(x)$, ALG uses at least two more wavelengths different from $w$ for the last $\lambda(x) + 1$ requests of $(x, v)$. Thus, ALG uses at least 3 wavelengths for the request sequence.

On the other hand, we have the following off-line algorithm. There is a set $\mathcal{A}$ of $\lambda(x)$ arc-disjoint $x$-arborescences in $G$ by Theorem II. For each of $\lambda(x)$ requests of $(x, v)$, we assign a dipath in distinct $x$-arborescence in $\mathcal{A}$, and assign the same wavelength, say $w$, to the dipaths. Since $\delta(x) > \lambda(x)$, there exists some $u_i$ $(1 \le i \le \delta(x))$ such that no dipaths above pass through $u_i$. Since $x$ is not a cut-vertex, there is a dipath $P(u_i, v)$ that dose not pass through $x$. For the remaining request of $(x, v)$, we assign a dipath consisting of arc $(x, u_i)$ and $P(u_i, v)$, and assign a wavelength different from $w$, say $w'$, to the dipaths. Then we can assign a dipath consisting of an arc $(x, u_j)$ with wavelength $w'$ for every requests $(x, u_j)$ $(j \ne i)$, and arc $(x, u_i)$ with wavelength $w$ for request $(x, u_i)$. In total, we use only 2 wavelengths for the request sequence, a contradiction. Thus we have the theorem. □

By corollary 2 and Theorem 3 above, we have the following corollary.

**Corollary 4.** *There is an on-line algorithm optimal for $G$ if and only if $\delta(x) = \lambda(x)$.*

We can show a general lower bound as follows. Let $M$ be a mesh with $V(M) = \{0, 1, 2\}^2$. The vertices $ij$ and $i'j'$ are adjacent if and only if $|i - i'| + |j - j'| = 1$. Let $G_M$ be the digraph associated with $M$.

**Theorem 4.** *The competitive ratio of any on-line algorithm for $G_M$ is at least 4/3.*

*Proof.* Let $u_1 = 01, u_2 = 10, u_3 = 12, u_4 = 21, v = 00$, and $x = 11$. Let ALG be any on-line algorithm for $G_M$. For any positive integer $l$, we consider the following sequence of $4l$ requests $I_{4l}$:

$$\underbrace{(x,u_1),\cdots,(x,u_1)}_{l},\underbrace{(x,u_2),\cdots,(x,u_2)}_{l},\underbrace{(x,u_3),\cdots,(x,u_3)}_{l},\underbrace{(x,u_4),\cdots,(x,u_4)}_{l}. \tag{4}$$

If $\omega(G_M, I_{4l}, \mathrm{ALG}) \geq 4l/3$ then we are done, because $\omega(G_M, I_{4l}) = l$ as easily seen, and we have

$$\omega(G_M, I_{4l}, \mathrm{ALG}) \geq \frac{4}{3}l = \frac{4}{3}\omega(G_M, I_{4l}).$$

If $\omega(G_M, I_{4l}, \mathrm{ALG}) < 4l/3$ then we consider the following sequence of additional $4l$ requests $I'_{4l}$:

$$\underbrace{(x,v),(x,v),\cdots,(x,v)}_{4l}. \tag{5}$$

Suppose that ALG uses $l+i$ ($0 \leq i < l/3$) wavelengths for the sequence (4), and let $W = \{w_1, w_2, \cdots, w_{l+i}\}$ be the set of wavelengths used for the sequence (4). Since the outdegree of $x$ is 4, the maximum number of requests for which we can assign wavelengths in $W$ is $4(l+i)$. Since the number of requests in the sequence (4) is $4l$, ALG can use the wavelengths in $W$ for at most $4(l+i) - 4l = 4i$ requests in the sequence (5). Since the indegree of $v$ is 2, ALG needs at least $(4l-4i)/2 = 2l-2i$ additional wavelengths not in $W$ for the sequence (5). Thus, ALG uses at least $(l+i)+(2l-2i) = 3l-i$ wavelengths for the concatenation of the sequences (4) and (5). Since $i < l/3$, we have

$$\omega(G_M, I_{4l} \cup I'_{4l}, \mathrm{ALG}) \geq 3l - i > 3l - \frac{1}{3}l = \frac{8}{3}l.$$

On the other hand, it is easy to see that $\omega(G_M, I_{4l} \cup I'_{4l}) = 2l$. Thus we have

$$\omega(G_M, I_{4l} \cup I'_{4l}, \mathrm{ALG}) > \frac{4}{3}\omega(G_M, I_{4l} \cup I'_{4l}),$$

as desired. □

Notice that $\omega(G_M, I, \mathrm{ARB}) \leq 2\omega(G_M, I)$ for any multicast $I$.

Our general upper bound for the competitive ratio is $\lceil \delta(x)/\lambda(x) \rceil$, and general lower bound is 4/3. It is an interesting open problem to close the gap between upper and lower bounds above.

## 4 Dynamic Multicasting

Given a symmetric digraph $G$ and a sequence of request arrivals and terminations for a multicast $I = \{(x,y) | y \in Y\}$, a dynamic algorithm assigns a dipath $P(x, y_i)$ and a wavelength to $P(x, y_i)$, so that no two dipaths sharing an arc are assigned the same wavelength if a request $(x, y_i)$ arrives, and deletes $P(x, y_i)$ together with the wavelength assigned if a request $(x, y_i)$ terminates.

Let $I_j$ denote a collection of the existing requests just after $j$th request arrival or termination in the sequence. We denote by $\omega(G, x, L, \mathrm{ALG}, I_j)$ the number of wavelengths used by a dynamic algorithm ALG for $I_j$ provided that $\mu(G, I_j) \leq L$ for any $j$. Let $\omega(G, x, L, \mathrm{ALG})$ denote $\max_j \omega(G, x, L, \mathrm{ALG}, I_j)$ and $\omega(G, x, L)$ denote the smallest $\omega(G, x, L, \mathrm{ALG})$ over all dynamic algorithms ALG. Notice that $\omega(G, x, L) \geq L$.

Our dynamic algorithm ARB′ is obtained from ARB by just adding an operation that when an existing request terminates, ARB′ deletes the dipath assigned for the request together with wavelength assigned. The following results are immediate from the corresponding results in the previous section.

**Theorem 5.**

$$\omega(G, x, L, \mathrm{ARB}') \leq \left\lceil \frac{L \cdot \delta(x)}{\lambda(x)} \right\rceil.$$

**Corollary 5.** *If* $\delta(x) = O(1)$ *then* $\omega(G, x, L, \mathrm{ARB}') = O(L)$.

**Theorem 6.** $\omega(G, x, L) = L$ *if and only if* $\delta(x) = \lambda(x)$.

**Theorem 7.**

$$\omega(G_M, x, L) \geq \frac{4}{3} L.$$

It should be noted that the performance of dynamic optical routing is considerably less than that of on-line optical routing in general, as mentioned in [6]. Our results indicate that the performance of dynamic multicasting is comparable to that of on-line multicasting.

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