LETTER

Optimal Layouts of Virtual Paths in Complete Binary Tree Networks

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SUMMARY It is a fundamental problem to construct a virtual path layout minimizing the hop number as a function of the congestion for a communication network. It is known that we can construct a virtual path layout with asymptotically optimal hop number for a mesh of trees network, butterfly network, cube-connected-cycles network, de Bruijn network, shuffle-exchange network, and complete binary tree network. The paper shows a virtual path layout with minimum hop number for a complete binary tree network. A generalization to complete k-ary tree networks is also mentioned.

key words: *communication network, complete binary tree network, congestion, hop number*

1. Introduction

We consider communication networks in which pairs of nodes exchange messages along pre-defined paths, called virtual paths. Each connection between two nodes must consist of a concatenation of such virtual paths. The layout is a collection of virtual paths that guarantees the connection for every pair of nodes. The hop number of a layout is the maximum, taken over all pairs of nodes, of the smallest number of virtual paths used to connect a pair of nodes. The congestion of a layout is the maximum number of virtual paths that pass through a link. The hop number corresponds to the time to set up a connection between a pair of nodes, and the congestion measures the load of the routing tables at the nodes.

It is a fundamental problem to construct a layout minimizing the hop number as a function of the congestion. For a network G, $\mathcal{H}_G(c)$ is the minimum hop number over all layouts with congestion at most c. Kranakis, Krizanc, and Pelc [2] showed a general lower bound for $\mathcal{H}_G(c)$. They proved that for any N-node network G with maximum node degree Δ, and for any positive integer c, $\mathcal{H}_G(c) \geq \log N/\log(c\Delta) - 1$. On the other hand, Stacho and Vrťo [3] showed a general layout with hop number $O(\mathrm{diam}(G)\log\Delta/\log c)$, where $\mathrm{diam}(G)$ is the diameter of G. It follows that if $\Delta = O(1)$ and $\mathrm{diam}(G) = O(\log N)$ then $\mathcal{H}_G(c) = \Theta(\log N/\log c)$ for any c. In particular, we have asymptotically optimal bounds for $\mathcal{H}_G(c)$ if G is a mesh of trees network, butterfly network, cube-connected-cycles network, de Bruijn network, shuffle-exchange network, and complete binary tree network. However, there is a considerable gap between the upper and lower bounds above. In fact, the constant factor hidden in the upper bound for complete binary tree networks is larger than 8.

The purpose of the paper is to close the gap for complete binary tree networks. We show the exact value of minimum hop number for complete binary tree networks. Our result is presented in the following theorem.

Theorem 1 For an N-node complete binary tree network B_N with height h and a positive integer c,

$$\mathcal{H}_{B_N}(c) = \begin{cases} 1 & \text{if } 1 \leq h \leq \left\lfloor \log\left(1+\sqrt{1+4c}\right)\right\rfloor - 1, \\ 2 & \text{if } \left\lfloor \log\left(1+\sqrt{1+4c}\right)\right\rfloor \leq h \leq \lfloor \log(c+1)\rfloor, \\ 3 + \left\lceil \frac{2h - 2\lfloor\log(c+1)\rfloor - \lceil\log c\rceil - 1}{\lfloor\log c\rfloor + 1}\right\rceil & \text{otherwise,} \end{cases}$$

where $h = \log(N+1) - 1$. □

2. Proof of Theorem 1

2.1 Upper Bounds

Case 1. $1 \leq h \leq \lfloor\log(1+\sqrt{1+4c})\rfloor - 1$:
Let $\mathcal{L}_1$ be a set of virtual paths connecting every pair of distinct vertices. $\mathcal{L}_1$ is a layout by definition.

Lemma 1: The congestion of layout $\mathcal{L}_1$ is at most c.

Proof: It is easy to see that the congestion is equal to the number of virtual paths in $\mathcal{L}_1$ that pass through an edge incident with the root. Hence, the congestion is bounded by

$$\begin{aligned} 2^h(2^h-1) &\leq 2^{\lfloor\log(1+\sqrt{1+4c})\rfloor - 1} \\ &\quad \times (2^{\lfloor\log(1+\sqrt{1+4c})\rfloor - 1} - 1) \\ &\leq \frac{\sqrt{1+4c}+1}{2}\,\frac{\sqrt{1+4c}-1}{2} \\ &= c. \end{aligned}$$

□

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Lemma 2: The hop number of layout $\mathcal{L}_1$ is 1.

Proof: Since every pair of distinct vertices is connected by a virtual path in $\mathcal{L}_1$, we have the lemma. $\square$

Case 2. $\lfloor \log(1+\sqrt{1+4c}) \rfloor \le h \le \lfloor \log(c+1) \rfloor$:
Let $\mathcal{L}_2$ be a set of virtual paths connecting the root of the tree and all the other vertices. It is easy to see that $\mathcal{L}_2$ is a layout.

Lemma 3: The congestion of layout $\mathcal{L}_2$ is at most c.

Proof: It is easy to see that the congestion is equal to the number of virtual paths in $\mathcal{L}_2$ that pass through an edge incident with the root. Thus, the congestion is bounded by

$$\begin{aligned} 2^h - 1 &\le 2^{\lfloor \log(c+1) \rfloor} - 1 \\ &\le 2^{\log(c+1)} - 1 \\ &\le c + 1 - 1 \\ &= c. \end{aligned}$$

$\square$

Lemma 4: The hop number of layout $\mathcal{L}_2$ is 2.

Proof: Any pair of distinct vertices is connected by a concatenation of at most two virtual paths in $\mathcal{L}_2$. Since a pair of distinct leaves is connected by a concatenation of two virtual paths in $\mathcal{L}_2$, we have the lemma. $\square$

Case 3. $h \ge \lfloor \log(c+1) \rfloor + 1$:
Recall that the level of vertices is recursively defined as follows: the level of the root is 0; if the level of a vertex is k then the level of a child of the vertex is $k+1$. Let m be the integer satisfying $h = \lfloor \log(c+1) \rfloor + m(\lfloor \log c \rfloor + 1) + x$, where $1 \le x \le \lfloor \log c \rfloor + 1$. Let $\mathcal{P}$ be a set of virtual paths connecting each vertex v on level $h - \lfloor \log(c+1) \rfloor$ and the descendants of v. For any integer $i (0 \le i \le m)$, let $\mathcal{Q}_i$ be a set of virtual paths connecting each vertex v on level $h - \lfloor \log(c+1) \rfloor - i(\lfloor \log c \rfloor + 1)$ and the ancestors of v on levels l for $h - \lfloor \log(c+1) \rfloor - (i+1)(\lfloor \log c \rfloor + 1) \le l \le h - \lfloor \log(c+1) \rfloor - i(\lfloor \log c \rfloor + 1) - 1$.

Subcase 3-1. $1 \le x \le (\lceil \log c \rceil + 1)/2$: Let $\mathcal{R}$ be a set of two virtual paths connecting the root r and two children s, t of r, $\mathcal{A}$ be a set of virtual paths connecting each vertex v on level x and the ancestors of v except r, and $\mathcal{B}$ be a set of virtual paths connecting all pairs of distinct vertices on level x. Define that $\mathcal{L}_{3-1} = \bigcup_{0 \le i \le m-1} \mathcal{Q}_i \cup \mathcal{R} \cup \mathcal{A} \cup \mathcal{B} \cup \mathcal{P}$. It is easy to see that $\mathcal{L}_{3-1}$ is a layout.

Subcase 3-2. $(\lceil \log c \rceil + 1)/2 < x \le \lfloor \log c \rfloor + 1$: Define that $\mathcal{L}_{3-2} = \bigcup_{0 \le i \le m} \mathcal{Q}_i \cup \mathcal{P}$. It is easy to see that $\mathcal{L}_{3-2}$ is a layout.

Lemma 5: The congestion of layouts $\mathcal{L}_{3-1}$ and $\mathcal{L}_{3-2}$ is at most c.

Proof: We prove the lemma by a series of claims.

Claim 1: There are not more than c virtual paths in $\mathcal{P}$ that share an edge.

Proof: The claim follows from Lemma 3, since $\mathcal{P}$ is just $\mathcal{L}_2$ for $h = \lfloor \log(c+1) \rfloor$. $\square$

Claim 2: There are not more than c virtual paths in $\mathcal{Q}_i$ that share an edge $(0 \le i \le m)$.

Proof: It is easy to see that the number of virtual paths in $\mathcal{Q}_i$ that pass through an edge incident with a vertex on the lowest level is maximal. Thus, the number of virtual paths in $\mathcal{Q}_i$ that share an edge is at most $2^{\lfloor \log c \rfloor} \le 2^{\log c} \le c$. $\square$

Claim 3: There are not more than c virtual paths in $\mathcal{R} \cup \mathcal{A} \cup \mathcal{B}$ that share an edge.

Proof: The number of virtual paths in $\mathcal{R} \cup \mathcal{A} \cup \mathcal{B}$ that pass through an edge incident with the root is at most

$$\begin{aligned} 2^{x-1} 2^{x-1} + 1 &= 2^{2x-2} + 1 \\ &\le 2^{\lceil \log c \rceil - 1} + 1 \\ &\le 2^{\log(c-1)} + 1 \\ &= c. \end{aligned}$$

The numbers of virtual paths in $\mathcal{R} \cup \mathcal{A} \cup \mathcal{B}$ that pass through any other edge is at most

$$\begin{aligned} & 2^{x-1} 2^{x-2} + 2^{x-2} 2^{x-2} + 2^{x-2} \\ &= 3 \times 2^{2x-4} + 2^{2x-2} \\ &\le 3 \times 2^{\lceil \log c \rceil - 3} + 2^{(\lceil \log c \rceil - 3)/2} \\ &\le \frac{3 \times 2^{\log(c-1)}}{4} + \frac{\sqrt{2^{(\log(c-1))}}}{2} \\ &= \frac{3(c-1)}{4} + \frac{\sqrt{c-1}}{2} \\ &\le \frac{3}{4}(c-1) + \frac{c}{4} < c. \end{aligned}$$

Thus, we have the claim. $\square$

The following claims are direct from the definition.

Claim 4: No virtual path in $\mathcal{P}$ shares an edge with a virtual path in $\bigcup_{0 \le i \le m} \mathcal{Q}_i \cup \mathcal{R} \cup \mathcal{A} \cup \mathcal{B}$.

Claim 5: No virtual path in $\mathcal{Q}_j$ shares an edge with a virtual path in $\bigcup_{0 \le i \le m} \mathcal{Q}_i \cup \mathcal{R} \cup \mathcal{A} \cup \mathcal{B} - \mathcal{Q}_j$.

From the claims above, we conclude that the congestion of $\mathcal{L}_{3-1}$ and $\mathcal{L}_{3-2}$ is at most c. This completes the proof of Lemma 5. $\square$

Lemma 6: The hop number of layouts $\mathcal{L}_{3-k}$ $(k = 1, 2)$ is at most

$$3 + \left\lceil \frac{2h - 2\lfloor \log(c+1) \rfloor - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil.$$

Proof: It is easy to see the following.

Claim 6: Let u and v be any vertices on levels $x + i(\lfloor \log c \rfloor + 1)$ and $x + j(\lfloor \log c \rfloor + 1)$, respectively. Then, u and v are connected by a concatenation of at most $i + j + k$ virtual paths in $\mathcal{L}_{3-k}$ $(k = 1, 2)$.

For any $v \in V(B_N) - \{r\}$, there exists a vertex $w \in V(B_N)$ on level $x + i(\lfloor \log c \rfloor + 1)$ for some integer i such that v and w are connected by a virtual path in $\mathcal{L}_{3-k}$ $(k = 1, 2)$. Hence, any two vertices $u, v \in V(B_N) - \{r\}$ are connected by a concatenation of at most $2m + 2 + k$ virtual paths in $\mathcal{L}_{3-k}$ $(k = 1, 2)$ by Claim 6. Since any vertex on level x and r are connected by a concatenation of at most two virtual paths, any $v \in V(B_N)$ and r are connected by a concatenation of at most $m + 3$ virtual paths in $\mathcal{L}_{3-k}$ $(k = 1, 2)$. Thus, the hop number of $\mathcal{L}_{3-k}$ $(k = 1, 2)$ is at most

$$2 + 2m + k = 2 + \frac{2h - 2\lfloor \log(c+1) \rfloor - 2x}{\lfloor \log c \rfloor + 1} + k.$$

Case 1. $k = 1$: Since $x \geq 1$, we have

$$\begin{aligned}
&2 + \frac{2h - 2\lfloor \log(c+1) \rfloor - 2x}{\lfloor \log c \rfloor + 1} + k \\
&\leq 2 + \left\lfloor \frac{2h - 2\lfloor \log(c+1) \rfloor - 2}{\lfloor \log c \rfloor + 1} \right\rfloor + 1 \\
&\leq 3 + \left\lceil \frac{2h - 2\lfloor \log(c+1) \rfloor - 2}{\lfloor \log c \rfloor + 1} - \frac{\lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil \\
&= 3 + \left\lceil \frac{2h - 2\lfloor \log(c+1) \rfloor - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil.
\end{aligned}$$

Case 2. $k = 2$: Since $x \geq (\lceil \log c \rceil + 2)/2$, we have

$$\begin{aligned}
&2 + \frac{2h - 2\lfloor \log(c+1) \rfloor - 2x}{\lfloor \log c \rfloor + 1} + k \\
&\leq 2 + \left\lfloor \frac{2h - 2\lfloor \log(c+1) \rfloor - \lceil \log c \rceil - 2}{\lfloor \log c \rfloor + 1} \right\rfloor + 2 \\
&= 4 + \left\lfloor \frac{2h - 2\lfloor \log \rfloor (c+1) \lceil \log \rceil c - 1}{\lfloor \log \rfloor c + 1} \right. \\
&\qquad \left. - \frac{1}{\lfloor \log c \rfloor + 1} \right\rfloor \\
&\leq 3 + \left\lceil \frac{2h - 2\lfloor \log(c+1) \rfloor - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil.
\end{aligned}$$

This completes the proof of Lemma 6. □

2.2 Lower Bounds

Case 1. $1 \leq h \leq \lfloor \log(1 + \sqrt{1 + 4c}) \rfloor - 1$:
$\mathcal{L}_1$ defined in Sect. 2.1.1 has hop number of 1 which is certainly optimal, and we have nothing to prove for this case.

Case 2. $\lfloor \log(1 + \sqrt{1 + 4c}) \rfloor \leq h \leq \lfloor \log(c + 1) \rfloor$:

Lemma 7: The hop number of any layout with congestion c is at least 2.

Proof: The hop number is 1 only when the layout is a set of virtual paths connecting every pair of distinct verices. It is easy to see that the number of virtual paths in such a layout that pass through an edge incident with the root is at least

$$\begin{aligned}
2^h(2^h - 1) &\geq 2^{\lfloor \log(1+\sqrt{1+4c}) \rfloor}(2^{\lfloor \log(1+\sqrt{1+4c}) \rfloor} - 1) \\
&> \frac{\sqrt{1+4c}+1}{2} \frac{\sqrt{1+4c}-1}{2} \\
&= c.
\end{aligned}$$

Thus, the congestion of such a layout is more than c, and we have the lemma. □

Case 3. $h \geq \lfloor \log(c + 1) + 1 \rfloor$:

Lemma 8: The hop number of any layout with congestion c is at least

$$3 + \left\lceil \frac{2h - 2\lfloor \log(c+1) \rfloor - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil.$$

Proof: We prove the lemma by a series of claims.

Claim 7: Let v be a vertex on level $h - \lfloor \log(c+1) \rfloor - 1$. Then, for any layout with congestion c, there exists a descendant u of v such that no virtual path in the layout starting at u contains v.

Proof: The proof is by contradiction. Suppose contrary that for each descendant w of v, there exists a virtual path that starts at w and contains v. Then the number of virtual paths that pass through an edge connecting v and a child of v is

$$\begin{aligned}
2^{\lfloor \log(c+1) \rfloor + 1} - 1 &= 2 \times 2^{\lfloor \log(c+1) \rfloor} - 1 \\
&> 2 \times \frac{c+1}{2} - 1 \\
&= c,
\end{aligned}$$

contradicting to the assumption that the congestion is c. □

Claim 8: Let v be a vertex on level $l \leq h - \lfloor \log c \rfloor - 2$. Then, for any layout with congestion c, there exists a descendant u of v on level $l + \lfloor \log c \rfloor + 2$ such that no virtual path in the layout contains both u and v.

Proof: The proof is by contradiction. Notice that the number of v's descendants on level $l + \lfloor \log c \rfloor + 2$ is $2^{\lfloor \log c \rfloor + 2}$. Suppose contrary that for each v's descendant w on level $l + \lfloor \log c \rfloor + 2$, there exists a virtual path that contains both w and v. Then the number of virtual paths that pass through an edge connecting v and a child of v is

$$2^{\lfloor \log c \rfloor + 1} = 2 \times 2^{\lfloor \log c \rfloor} > 2 \times \frac{c}{2} = c,$$

contradicting to the assumption that the congestion is c. □

Claim 9: For any layout with congestion c, if $x > (\lceil \log c \rceil + 1)/2$ then there exists a pair of vertices u and v on level x such that no virtual path contains both u and v.

Proof: The proof is by contradiction. Notice that if s and t are the children of the root, the number of pairs of s's descendant and t's descendant on level x is at least $2^{2x-2} \geq 2^{\lceil \log c \rceil} \geq c$. Suppose contrary that for each such pair of u and v, there exists a virtual path that contains both u and v. By the definition of virtual path layout, there exists at leaset one virtual path starting at the root. Then the number of virtual paths that pass through an edge incident with the root is at least $c+1$, contradicting to the assumption that the congestion is c. □

From the claims above, we conclude that there exists a pair of s's descendant $\bar{s}$ on level l, $h - \lfloor \log (c+1) \rfloor < l \leq h$, and t's descendant $\bar{t}$ on level l', $h - \lfloor \log (c+1) \rfloor < l' \leq h$, such that at least

$$
\begin{aligned}
2+2m+&\left\lceil \frac{2x}{\lceil \log c \rceil + 1} \right\rceil \\
&= 2+\frac{2h-2\lfloor \log (c+1) \rfloor - 2x}{\lfloor \log c \rfloor + 1} + \left\lceil \frac{2x}{\lceil \log c \rceil + 1} \right\rceil \\
&= 2+\frac{2h-2\lfloor \log (c+1) \rfloor - 2x}{\lfloor \log c \rfloor + 1} \\
&\quad + \left\lceil \frac{2x - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil + 1 \\
&= 3+\left\lceil \frac{2h-2\lfloor \log (c+1) \rfloor - \lceil \log c \rceil - 1}{\lfloor \log c \rfloor + 1} \right\rceil
\end{aligned}
$$

virtual paths are needed to connect $\bar{s}$ and $\bar{t}$. Here we used that $h = \lfloor \log(c+1) \rfloor + m(\lfloor \log c \rfloor + 1) + x$, and $1 \leq x \leq \lfloor \log c \rfloor + 1$. This completes the proof of Lemma 8. □

3. Concluding Remarks

1. Theorem 1 can be generalized to complete k-ary tree networks as follows.

Theorem 2 For an N-node complete k-ary tree network $T_{k,N}$ with height h and a positive integer c,

$$
\mathcal{H}_{T_{k,N}}(c) =
\begin{cases}
1 \quad \text{if} \quad 1 \leq h \leq \left\lfloor \log_k \left(1+\sqrt{1+4c(k-1)}\right) - \log_k 2 \right\rfloor, \\
2 \quad \text{if} \quad h \leq \lfloor \log_k((k-1)c+1) \rfloor \quad \text{and} \\
\qquad\quad h \geq \left\lfloor \log_k \left(1+\sqrt{1+4c(k-1)}\right) - \log_k 2 \right\rfloor + 1, \\
3+\left\lceil \frac{2h-2\lfloor \log_k((k-1)c+1) \rfloor - \lceil \log_k \frac{c}{k-1} \rceil - 1}{\lfloor \log_k c \rfloor + 1} \right\rceil \quad \text{otherwise,}
\end{cases}
$$

where $h = \log_k((k-1)N+1) - 1$. □

2. The corresponding problem for directed graphs is considered in [1].

References

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