ON THREE-DIMENSIONAL LAYOUT OF DE BRUIJN NETWORKS

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ABSTRACT

The de Bruijn networks are well-known as suitable structures for parallel computations such as FFT. This paper shows an efficient 3D VLSI layout of the de Bruijn network with optimal volume and near optimal wire-length. Our layout consists of a number of copies of a single 2D VLSI layout for a subnetwork of the de Bruijn network.

1. INTRODUCTION

There has been a great interest in embedding graphs into 3D(three-dimensional) grids motivated by 3D VLSI circuits and 3D drawings. This paper is concerned with 3D layouts of de Bruijn networks, which are well-known as suitable structures for parallel computations such as FFT[8, 11].

The notion of 3D layout of a VLSI circuit follows the classic 2D framework. The circuit is represented by a graph and the media in which the circuit is to be realized is a 3D rectangular grid. A circuit layout is a vertex-disjoint embedding of the circuit-graph in the grid. The cost of a layout is measured by the volume and wire-length of the layout. It follows from general lower bounds derived by Rosenberg[10] that the volume and wire-length of a 3D layout for an N-vertex de Bruijn network are $\Omega(N^{3/2}/\log^{3/2} N)$ and $\Omega(N^{1/2}/\log^{3/2} N)$, respectively. On the other hand, it is implicit in [5] by Kock, Leighton, Maggs, Rao, Rosenberg, and Schwabe that an N-vertex de Bruijn network can be laid out in $O(N^{3/2}/\log^{3/2} N)$ volume. However, the layout is indirect and complicated in the sense that the layout is based on a 3D layout of a butterfly network by Wise[13] via a shuffle-exchange network.

This paper shows direct and simple 3D layouts of de Bruijn networks. The volume of our layout is optimal, and the wire-length is close to the optimal. More precisely, the volume and wire-length of our layout for an N-vertex de Bruijn network are $O(N^{3/2}/\log^{3/2} N)$ and $O(N^{1/2}/\log^{1/2} N)$, respectively. Our layout is based on the VLSI decomposition of de Bruijn networks extensively studied in the literature in connection with the construction of large Viterbi decoders[1, 2, 3, 6, 12, 14]. A VLSI decomposition of a de Bruijn network is a collection of isomorphic vertex-disjoint subnetworks (called building blocks) which together span the de Bruijn network. That is, a de Bruijn network can be built by wiring together the same building blocks. The efficiency of such a building block is defined as the fraction of the edges of a de Bruijn network which are present in the copies of the building block. We use an (asymptotically) optimal building block proposed by Schwabe[12]. We lay out the optimal building block in a 2-layer rectangular grid, which represents a printed circuit board or a VLSI chip. Such 2-layer rectangular grids are put one atop another, and wired together to lay out a de Bruijn network in a 3D rectangular grid. (See Fig. 3.)

The proofs are omitted in the extended abstract due to space limitation.

2. PRELIMINARIES

2.1. De Bruijn Networks Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. We denote by $\delta_G(v)$ the degree of a vertex $v \in V(G)$ and define $\Delta(G) = \max\{\delta_G(v) : v \in V(G)\}$.

Let $[m] = \{0, 1, \ldots, m-1\}$ for any positive integer m. $[2]^n$ is the set of binary strings of length n, and $\bar{x}$ is the binary complement of $x \in [2]$. We define two bijections on $[2]^n$ as follows: $\sigma(v_1 v_2 \cdots v_n) = v_2 \cdots v_n v_1$; $\gamma(v_1 v_2 \cdots v_n) = v_2 \cdots v_n \overline{v_1}$.

The n-dimensional de Bruijn graph, denoted by $dB(n)$, is the graph defined as follows: $V(dB(n)) = [2]^n$; $E(dB(n)) = \{(u,v) : v = \sigma(u) \text{ or } u = \sigma(v)\} \cup \{(u,v) : v = \gamma(u) \text{ or } u = \gamma(v)\}$. $dB(n)$ consists of $N = 2^n$ vertices and $2N$ edges.

2.2. Layouts An embedding $\langle \phi, \rho \rangle$ of a graph G into a graph H consists of a one-to-one mapping $\phi : V(G) \to V(H)$, together with a mapping ρ that maps each edge $(u,v) \in E(G)$ onto a path $\rho(u,v)$ in H that connects $\phi(u)$ and $\phi(v)$. The congestion of $\langle \phi, \rho \rangle$ is defined as $\max_{e \in E(H)} |\{e' \in E(G) : \rho(e') \text{ contains } e\}|$.

The d-dimensional $m_1 \times m_2 \times \cdots \times m_d$ grid, denoted by $R(m_1, m_2, \ldots, m_d)$, is the graph defined as follows: $V(R(m_1, m_2, \ldots, m_d)) = [m_1] \times [m_2] \times \cdots \times [m_d]$; $E(R(m_1, m_2, \ldots, m_d)) = \{(u,v) : \sum_{i=1}^{d} |v_i - u_i| = 1\}$, where $u = [u_1, u_2, \ldots, u_d]$ and $v = [v_1, v_2, \ldots, v_d]$.

A layout of a graph G into a 2-dimensional grid R is an embedding $\langle \phi, \rho \rangle$ of G into R with congestion 1 and no knock-knee paths. A layout of a graph G into a 3-dimensional grid R is an embedding $\langle \phi, \rho \rangle$ of G into R such that $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint for any distinct $e_1, e_2 \in E(G)$. $\langle \phi, \rho \rangle$ is called a 3-D[2-D] layout of G if $\langle \phi, \rho \rangle$ is a layout of G into a 3-dimensional[2-dimensional] grid R. The volume[area] of a 3-D[2-D] layout $\langle \phi, \rho \rangle$ is the number of vertices in R. The wire-length of $\langle \phi, \rho \rangle$ is the maximum length of a path $\rho(e)$.

2.3. VLSI Decompositions of de Bruijn Networks For any graph G and positive integer k, let kG denote a disjoint union of k copies of G. If kG is a spanning subgraph of a graph H, kG and G are called a VLSI decomposition and a building block for H, respectively. A spanning subgraph $G(m)$ of $dB(m)$ is called a

0-7803-7448-7/02/$17.00

universal de Bruijn building block of order m if $G(m)$ is a building block of $dB(m+r)$ for any natural number r. The following is due to Schwabe.

Theorem I [12] *For any positive integers m, we can construct a universal de Bruijn building block $BS(m)$ of order m such that* $|E(dB(m))| - |E(BS(m))| \leq 2^{m+4}/(m+4)$. ∎

Since it is known that $|E(dB(m))| - |E(G(m))|$ is $\Omega(2^m/m)$ for any universal de Bruijn building block $G(m)$ [12], $BS(m)$ is optimal in the sense that $BS(m)$ has an asymptotically maximal number of edges. Since $2^r BS(m)$ is a spanning subgraph of $dB(m+r)$, $V(dB(m+r))$ can be partitioned into 2^r subsets $(V_0, V_1, \ldots, V_{2^r-1})$ of size 2^m in such a way that there exists a bijection $\mu_i : V(BS(m)) \to V_i$ such that $(\mu_i(u), \mu_i(v)) \in E(dB(m+r))$ if $(u,v) \in E(BS(m))$ $(i \in [2^r])$. $(V_0, V_1, \ldots, V_{2^r-1})$ is called an S-partition of $V(dB(m+r))$, and μ_i is called an S-mapping.

3. 1-D EMBEDDINGS

Raspaud, Sýkora, and Vrt'o showed the existence of an embedding of $dB(n)$ into $R(2^n)$ with congestion $O(2^n/n)$ [9]. In this section, we show such an embedding explicitly.

We recursively define an embedding $\langle \phi_{1,n}, \rho_{1,n} \rangle$ of $dB(n)$ into $R(2^n)$ as follows: (i) $\phi_{1,1}(v) = v$ for any $v \in [2]$. (ii) If (V_0, V_1) is an S-partition of $V(dB(n))$ and μ_i is an S-mapping $(i \in [2])$, $\phi_{1,n}(v) = i2^{n-1} + \phi_{1,n-1}(\mu_i^{-1}(v))$ if $v \in V_i$. $\rho_{1,n}(e)$ is defined as the unique path in $R(2^n)$ connecting $\phi_{1,n}(u)$ and $\phi_{1,n}(v)$ for any edge $e = (u,v) \in E(dB(n))$ $(n \geq 1)$.

We can prove the following.

Theorem 1 *$\langle \phi_{1,n}, \rho_{1,n} \rangle$ is an embedding of $dB(n)$ into $R(2^n)$ with congestion $c(n) \leq (1 + 3/(n+3)) \cdot 2^{n+5}/(n+3)$.* ∎

4. 2-D LAYOUTS

Samatham and Pradhan proved that $dB(n)$ can be laid out in a 2-dimensional grid with area $O(2^{2n}/n^2)$ [11]. However, the layout is indirect and complicated in the sense that the layout is based on a 2-D layout of a shuffle-exchange graph by Leighton[7]. In this section, we present a direct and simple 2-D layout of $dB(n)$ with area $O(2^{2n}/n^2)$. For simplicity, we show a 2-D layout of $dB(n)$ for even n.

Let $\mathcal{N}$ be the set of natural numbers. A scheduling of $E(dB(n))$ is a mapping $f : E(dB(n)) \to \mathcal{N}$ such that $f(e) \neq f(e')$ if $\rho_{1,n}(e)$ and $\rho_{1,n}(e')$ share a vertex. A scheduling f of $E(dB(n))$ is called a k-scheduling if $f(E(dB(n))) \subseteq [k]$.

For $e = (u,v) \in E(dB(n))$, we define that $\min(e) = \min\{\phi_{1,n}(u), \phi_{1,n}(v)\}$. Suppose that $E(dB(n)) = \{e_i : i \in [2^{n+1}]\}$ and $i < j$ if $\min(e_i) < \min(e_j)$. We define a mapping $\psi_n : E(dB(n)) \to \mathcal{N}$ as follows: $\psi_n(e_0) = 0$; $\psi_n(e_i)$ is the smallest $l \in \mathcal{N}$ such that $\psi_n(e_j) \neq l$ if $j < i$, and $\rho_{1,n}(e_j)$ and $\rho_{1,n}(e_i)$ share a vertex. We can prove that ψ_n is a $(c(n)+2)$-scheduling of $E(dB(n))$. Let $u = u_1 \cdots u_n$, $v = v_1 \cdots v_n \in [2]^n$, and let $u_H = u_1 \cdots u_{n/2}$, $u_L = u_{(n/2)+1} \cdots u_n$, $v_H = v_1 \cdots v_{n/2}$, and $v_L = v_{(n/2)+1} \cdots v_n$. It is easy to see that $(u_H, v_H), (u_L, v_L) \in E(dB(n/2))$ if $(u,v) \in E(dB(n))$.

Now, we are ready to define a layout $\langle \phi_{2,n}, \rho_{2,n} \rangle$ of $dB(n)$ into $R(c_n 2^{n/2}, c_n 2^{n/2})$, where c_n is an integer at least $c(n/2)+4$. We define that $\phi_{2,n}(v) = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [1,1]$.

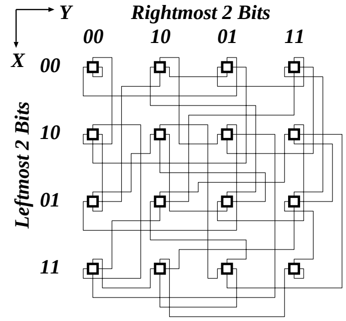

Fig. 1. 2-D layout $\langle \phi_{2,2}, \rho_{2,2} \rangle$ of $dB(4)$ into $R(28,28)$.

It remains to define $\rho_{2,n}$. For any $e = (u,v) \in E(dB(n))$ with $v = \sigma(u)$, define that: $w_1 = \phi_{2,n}(u) = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(u_L)] + [1,1]$; $w_2 = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(u_L)] + [\psi_{n/2}(u_L, v_L) + 2, 1]$; $w_3 = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(v_L)] + [\psi_{n/2}(u_L, v_L) + 2, \psi_{n/2}(u_H, v_H) + 2]$; $w_4 = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [1, \psi_{n/2}(u_H, v_H) + 2]$; $w_5 = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [1,1] = \phi_{2,n}(v)$. For any $e = (u,v) \in E(dB(n))$ with $v = \gamma(u)$, define that: $w'_0 = \phi_{2,n}(u) = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(u_L)] + [1,1]$; $w'_1 = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(u_L)] + [1,0]$; $w'_2 = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(u_L)] + [\psi_{n/2}(u_L, v_L) + 2, 0]$; $w'_3 = c_n \cdot [\phi_{1,n/2}(u_H), \phi_{1,n/2}(v_L)] + [\psi_{n/2}(u_L, v_L) + 2, \psi_{n/2}(u_H, v_H) + 2]$; $w'_4 = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [0, \psi_{n/2}(u_H, v_H) + 2]$; $w'_5 = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [0,1]$; $w'_6 = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L)] + [1,1] = \phi_{2,n}(v)$. Let P_i $[P'_j]$ denote the shortest path in $R(c_n 2^{n/2}, c_n 2^{n/2})$ connecting w_i and w_{i+1} $[w'_j$ and $w'_{j+1}]$ for $i \in \{1,2,3,4\}$ $[j \in [6]]$. For $e = (u,v) \in E(dB(n))$, $\rho_{2,n}(e)$ is defined as a concatenation of paths P_1, P_2, P_3, and P_4 if $v = \sigma(u)$, and as a concatenation of paths P'_0, P'_1, P'_2, P'_3, P'_4, and P'_5 if $v = \gamma(u)$. Fig.1 shows $\langle \phi_{2,2}, \rho_{2,2} \rangle$.

Similarly, we can define the layout $\langle \phi_{2,n}, \rho_{2,n} \rangle$ for odd n, and prove the following for every natural number n.

Theorem 2 *$\langle \phi_{2,n}, \rho_{2,n} \rangle$ is a layout of $dB(n)$ into $R(c_n 2^{n/2}, c_n 2^{n/2})$ with wire-length at most $2c_n(2^{n/2}+1)$.* ∎

It is easy to see that the area and wire-length of $\langle \phi_{2,n}, \rho_{2,n} \rangle$ are $O(2^{2n}/n^2)$ and $O(2^n/n)$ if $c_n = c(n/2) + 4$.

5. 3-D LAYOUTS

5.1. 2-Layer Layouts Let R_j be a $c_n 2^{n/2} \times c_n 2^{n/2}$ subgrid of $R(c_n 2^{n/2}, c_n 2^{n/2}, 2)$ induced by $\{[x,y,j] : x, y \in [c_n 2^{n/2}]\}$ for $j \in [2]$, and let $[x_i, y_i] = w_i$ and $[x'_i, y'_i] = w'_i$ for $i \in \{2,3,4,5\}$. A layout $\langle \varphi_{2,n}, \varrho_{2,n} \rangle$ of $dB(n)$ into $R(c_n 2^{n/2}, c_n 2^{n/2}, 2)$ is defined as follows: $\varphi_{2,n}(v) = c_n \cdot [\phi_{1,n/2}(v_H), \phi_{1,n/2}(v_L), 0] + [1,1,0]$; $\varrho_{2,n}(e)$ is the path obtained from $\rho_{2,n}(e)$ by combining two paths on R_1 corresponding to P_2 and P_4 $[P'_2$ and $P'_4]$ in $R(c_n 2^{n/2}, c_n 2^{n/2})$, paths on R_0 corresponding to the rest of $\rho_{2,n}(e)$ in $R(c_n 2^{n/2}, c_n 2^{n/2})$, and four edges $([x_i, y_i, 0], [x_i, y_i, 1])[([x'_i, y'_i, 0], [x'_i, y'_i, 1])]$ $(i \in \{1,2,3,4\})$ if $e = (u,v)$ with $v = \sigma(u)[v = \gamma(u)]$.

The following is direct from Theorem 2.

Theorem 3 $\langle \varphi_{2,n}, \varrho_{2,n} \rangle$ *is a layout of* $dB(n)$ *into* $R(c_n 2^{n/2}, c_n 2^{n/2}, 2)$ *with wire-length at most* $2c_n(2^{n/2}+1)+4$. ∎

5.2. 2-Layer Layout of $BS(m)$ Let $m = \lceil (n+\log n)/2 \rceil$ and $r = n - m$. Since $BS(m)$ is a spanning subgraph of $dB(m)$, the following is direct from Theorem 3.

Theorem 4 $\langle \varphi_{2,m}, \bar{\varrho}_{2,m} \rangle$ *is a layout of* $BS(m)$ *into* $R(c_m 2^{m/2}, c_m 2^{m/2}, 2)$ *with wire-length at most* $2c_m(2^{m/2}+1)+4$. *where* $\bar{\varrho}_{2,m}$ *is the restriction of* $\varrho_{2,m}$ *to* $E(BS(m))$. ∎

Let $(V_0, V_1, \ldots, V_{2^r-1})$ be an S-partition of $V(dB(n))$ and let $\mu_i : V(BS(m)) \to V_i$ be an S-mapping ($i \in [2^r]$). We define that $E_\mu = E(dB(n)) - \bigcup_{i=0}^{2^r-1}\{(\mu_i(u), \mu_i(v)) : (u,v) \in E(BS(m))\}$. $BS'(m)$ is the graph obtained from $BS(m)$ by adding $4 - \delta_{BS(m)}(v)$ vertices adjacent to each $v \in V(BS(m))$. $BS''(m)$ is the graph obtained from $BS'(m)$ by adding a vertex adjacent to each $v \in V(BS'(m))$ of degree 1. $dB''(n)$ is the graph obtained from $dB(n)$ by replacing each edge of E_μ with a path of length 5. If M is the set of middle edges of paths of length 5 corresponding to the edges of E_μ, $dB''(n) - M$ is isomorphic to $2^r BS''(m)$.

5.3. 4-Layer Layout of $BS'(m)$ A layout $\langle \varphi'_{2,m}, \varrho'_{2,m} \rangle$ of $BS'(m)$ into $R(c_m 2^{m/2}, c_m 2^{m/2}, 4)$ is defined as follows: $\varphi'_{2,m}(v) = \varphi_{2,m}(v) + [0,0,1]$ if $v \in V(BS(m))$; $\varrho'_{2,m}(e)$ is a path obtained from $\bar{\varrho}_{2,m}(e)$ by replacing each edge $([x,y,z], [x',y',z'])$ in $\bar{\varrho}_{2,m}(e)$ with an edge $([x,y,z+1], [x',y',z'+1])$ if $e \in E(BS(m))$. In order to complete the definition of $\langle \varphi'_{2,m}, \varrho'_{2,m} \rangle$, we need to define $\varphi'_{2,m}(v)$ for $v \in V(BS'(m)) - V(BS(m))$ and $\varrho'_{2,m}(e)$ for $e \in E(BS'(m)) - E(BS(m))$. If u is a vertex with $\delta_{BS(m)}(u) < 4$ then there exists d vertices $v_0, \ldots, v_{d-1} \in V(BS'(m)) - V(BS(m))$ adjacent to u in $BS'(m)$, where $d = 4 - \delta_{BS(m)}(u)$. Let $e_i = (u, v_i)$ for $i \in [d]$. If $\varphi'_{2,m}(u) = [x,y,1]$ then d of the following 4 paths are internally vertex-disjoint to $\varrho'_{2,m}(e)$ for every $e \in E(BS(m))$: $([x,y,1],[x,y,0])$, $([x,y,1],[x-1,y,1],[x-1,y,0])$, $([x,y,1],[x+1,y,1],[x+1,y,0])$, and $([x,y,1],[x,y,2],[x,y,3])$. By using d of these paths, we define $\varrho'_{2,m}(e_i)$ for $i \in [d]$. $\varphi'_{2,m}(v_i)$ is defined as the other endvertex of $\varrho'_{2,m}(e_i)$ than $[x,y,1] (= \varphi'_{2,m}(u))$.

We can prove the following.

Theorem 5 $\langle \varphi'_{2,m}, \varrho'_{2,m} \rangle$ *is a layout of* $BS'(m)$ *into* $R(c_m 2^{m/2}, c_m 2^{m/2}, 4)$ *with wire-length at most* $2c_m(2^{m/2}+1)+4$. ∎

Notice that every vertex with degree 1 is placed at a vertex $[x, c_m j+1, z]$ for some $x, j \in [2^{m/2}]$ and $z \in \{0,3\}$, and each of the other vertices is placed at a vertex $[x, c_m j+1, 1]$ for some x and $j \in [2^{m/2}]$.

5.4. 4-Layer Layout of $BS''(m)$ A layout $\langle \varphi''_{2,m}, \varrho''_{2,m} \rangle$ of $BS''(m)$ is defined as follows. Let $c_m = c(m/2)+4$ if $c(m/2)$ is even, and $c_m = c(m/2)+5$ otherwise. It should be noted that c_m is even. For $i \in [2^{m/2}]$, let $U_i = \{v \in V(BS'(m)) : \varphi'_{2,m}(v) = [x, c_m i+1, 0]\}$, $V_i = \{v \in V(BS'(m)) : \varphi'_{2,m}(v) = [x, c_m i+1, 3]\}$, $W_i = \{v \in V(BS'(m)) : \varphi'_{2,m}(v) = [x, c_m i+1, z]\}$, $k_i = |U_i|$, and $l_i = |V_i|$, and define that $a_i = \max\{2k_i - c_m, 2l_i - c_m, 0\}$ and $A_i = \sum_{j=0}^{i-1} a_j$. Notice that a_i is even, and so A_i is also even for any i since c_m is even. We define $\varphi''_{2,m}(v)$ and $\varrho''_{2,m}(e)$ for $v \in V(BS'(m))$ and $e \in E(BS'(m))$ as follows. $\varphi''_{2,m}(v) = \varphi'_{2,m}(v) + [0, A_i, 0]$ if $v \in W_i$ for some i. We define $\varrho''_{2,m}(e)$ as the path obtained from $\varrho'_{2,m}(e)$ by replacing each edge $([x,y,z],[x',y',z'])$ in $\varrho'_{2,m}(e)$ with an edge $([x, y+A_i, z], [x', y'+A_i, z'])$ if $c_m i \le y, y' \le c_m(i+1)-1$ for some i, and with a shortest path between $[x, y+A_i, z]$ and $[x', y'+A_{i+1}, z']$ if $y = c_m(i+1)-1$ and $y' = c_m(i+1)$ for some i. Let $U_i = \{u_{i,0}, \ldots, u_{i,k_i-1}\}$ and $V_i = \{v_{i,0}, \ldots, v_{i,l_i-1}\}$ in decreasing order of the 1st coordinates of $\varphi'_{2,m}(u)$ and $\varphi'_{2,m}(u)$, respectively. Let $u'_{i,j}$ and $v'_{i,j}$ denote the vertices adjacent to $u_{i,j}$ and $v_{i,j}$ in $BS''(m)$ such that $u'_{i,j}, v'_{i,j} \notin V(BS'(m))$, respectively. We define that $\varphi''_{2,m}(u'_{i,j}) = [c_m 2^{m/2}, c_m i + A_i + 2j + 1, 1]$ and $\varphi''_{2,m}(v'_{i,j}) = [c_m 2^{m/2}, c_m i + A_i + 2j + 1, 3]$. $\varrho''_{2,m}(u_{i,j}, u'_{i,j})$ is defined as the concatenation of the shortest path between $[x, c_m i + A_i + 1, 0]$ and $[x, c_m i + A_i + 2j + 1, 0]$, the shortest path between $[x, c_m i + A_i + 2j + 1, 0]$ and $[c_m 2^{m/2}, c_m i + A_i + 2j + 1, 0]$, and an edge $([c_m 2^{m/2}, c_m i + A_i + 2j + 1, 0], [c_m 2^{m/2}, c_m i + A_i + 2j + 1, 1])$, and $\varrho''_{2,m}(v_{i,j}, v'_{i,j})$ is defined as the concatenation of the shortest path between $[x, c_m i + A_i + 1, 3]$ and $[x, c_m i + A_i + 2j + 1, 3]$ and the shortest path between $[x, c_m i + A_i + 2j + 1, 3]$ and $[c_m 2^{m/2}, c_m i + A_i + 2j + 1, 3]$. (See Figure 2.)

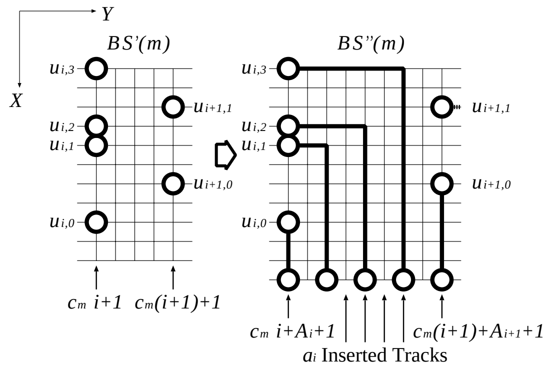

Fig. 2. Placement of vertices with degree 1 in $BS''(n)$.

We can prove the following.

Theorem 6 $\langle \varphi''_{2,m}, \varrho''_{2,m} \rangle$ *is a layout of* $BS''(m)$ *into* $R(c_m 2^{m/2}+1, c_m 2^{m/2}+A_{2^{m/2}}, 4)$ *with wire-length at most* $2c_m(2^{m/2}+1) + \max_i a_i + 4$. ∎

Notice that $\varphi''_{2,m}(v) = [c_m 2^{m/2}, 2p-1, 2q-1]$ for some positive integers p and q if $\delta_{BS''(m)}(v) = 1$.

5.5. 3-D Layouts Now, we are ready to define a 3-D layout $\langle \phi_{3,n}, \rho_{3,n} \rangle$ of $dB''(n)$. We denote 2^r copies of $BS''(m)$ by $BS''_0(m), \ldots$, and $BS''_{2^r-1}(m)$. For $i \in [2^r]$, we lay out $BS''_i(m)$ into $(c_m 2^{m/2}+1) \times (c_m 2^{m/2} + A_{2^{m/2}}) \times 4$ subgrid induced by

$$\left\{ [x, y, 4i+k] : \begin{array}{l} x \in [c_m 2^{m/2}+1],\ k \in [4] \\ y \in [c_m 2^{m/2} + A_{2^{m/2}}], \end{array} \right\}$$

using $\langle \varphi''_{2,m}, \varrho''_{2,m} \rangle$ for $BS''(m)$. We complete the definition of $\langle \phi_{3,n}, \rho_{3,n} \rangle$ by assigning a vertex-disjoint path to each $(u,v) \in M$. Let $V(M) = \{u, v : (u,v) \in M\}$. Notice that every vertex of $V(M)$ is placed at a vertex $[c_m 2^{m/2}, 2j+1, 2k+1]$ for some $j \in [(c_m 2^{m/2} + A_{2^{m/2}})/2]$ and $k \in [2^{r+1}]$. Let (S,T) be a partition of $V(M)$ such that either $u \in S$ and $v \in T$ or $u \in T$

and $v \in S$ for any $(u, v) \in M$. A bipartite (multi)graph B_M with bipartition $U(B_M)$ and $W(B_M)$ is defined as follows: $U(B_M) = \{u_j : j \in [(c_m 2^{m/2} + A_{2^{m/2}})/2]\}$; $W(B_M) = \{w_k : k \in [2^{r+1}]\}$. Any two vertices $u_{j_1} \in U(B_M)$ and $w_{k_2} \in W(B_M)$ are joined by

$$\left| \left\{ (s,t) : \begin{array}{l} s \in S,\ t \in T,\ (s,t) \in M \\ \phi_{3,n}(s) = [c_m 2^{m/2}, 2j_1 + 1, 2k_1 + 1], \\ \phi_{3,n}(t) = [c_m 2^{m/2}, 2j_2 + 1, 2k_2 + 1] \end{array} \right\} \right|$$

parallel edges. Notice that there is a one-to-one correspondence between the edges in M and edges in B_M. Let ϵ_e denote the edge in B_M corresponding to $e \in M$.

For a graph G, a mapping $\kappa : E(G) \to \mathcal{N}$ is an edge-coloring of G if $\kappa(e_1) \neq \kappa(e_2)$ for any adjacent edges $e_1, e_2 \in E(G)$. An edge-coloring of G is called a k-edge-coloring if $\kappa(E(G)) \subseteq [k]$. It is well-known that there exists a $\Delta(G)$-edge-coloring of G for any bipartite (multi)graph G.

Let f denote a $\Delta(B_M)$-edge-coloring of B_M. For each $e = (s,t) \in M$, define that: $v_0 = [c_m 2^{m/2}, 2j_1 + 1, 2k_1 + 1]$ $(= \phi_{3,n}(s))$; $v_1 = [c_m 2^{m/2} + 2f(\epsilon_e) + 1, 2j_1 + 1, 2k_1 + 1]$; $v_2 = [c_m 2^{m/2} + 2f(\epsilon_e) + 1, 2j_1, 2k_1 + 1]$; $v_3 = [c_m 2^{m/2} + 2f(\epsilon_e) + 1, 2j_1, 2k_2]$; $v_4 = [c_m 2^{m/2} + 2f(\epsilon_e) + 2, 2j_1, 2k_2]$; $v_5 = [c_m 2^{m/2} + 2f(\epsilon_e) + 2, 2j_2 + 1, 2k_2]$; $v_6 = [c_m 2^{m/2} + 2f(\epsilon_e) + 2, 2j_2 + 1, 2k_2 + 1]$; $v_7 = [c_m 2^{m/2}, 2j_2 + 1, 2k_2 + 1]$ $(= \phi_{3,n}(t))$. Let Q_i be the shortest path in $R(m_1, m_2, m_3)$ connecting v_i and v_{i+1} for $i \in [7]$, where $m_1 = c_m 2^{m/2} + 2\Delta(B_M) + 1$, $m_2 = c_m 2^{m/2} + A_{2^{m/2}}$, and $m_3 = 2^{r+2}$. We define for $e \in M$ that $\rho_{3,n}(e)$ is the concatenation of these seven paths, that is $\rho_{3,n}(e) = Q_0 Q_1 \cdots Q_6$. (See Fig.3.)

We can prove the following.

Theorem 7 *$\langle \phi_{3,n}, \rho_{3,n} \rangle$ is a layout of dB$''(n)$ into $R(m_1, m_2, m_3)$ with wire-length at most $2m_1 + m_2 + m_3 + 2(c_m + \max_i a_i)$.* ∎

Since $\langle \phi_{3,n}, \rho_{3,n} \rangle$ naturally induces a layout of *dB*(n), and $\Delta(B_M) = O(\sqrt{N/\log N})$ and $A_{2^{m/2}} = O(\sqrt{N/\log N})$ by Theorem I, we have the following:

Theorem 8 *We can construct a 3-D layout of dB(n) with volume $O(N^{3/2}/\log^{3/2} N)$ and wire length $O(\sqrt{N/\log N})$, where $N = 2^n$.* ∎

Acknowledgments: The research is a part of CAD21 Project at TIT.

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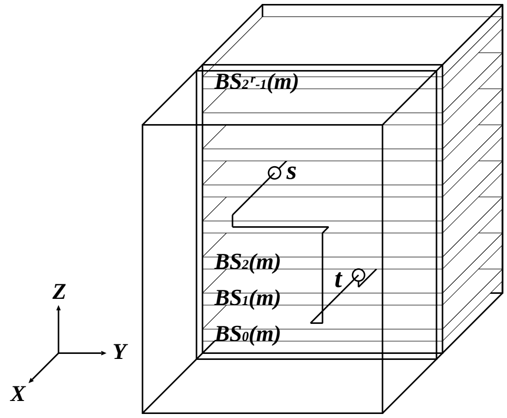

Fig. 3. 3-D Layout of *dB*(n).

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