ON THREE-DIMENSIONAL LAYOUT OF PYRAMID NETWORKS

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ABSTRACT

The pyramid networks are well-known as suitable structures for parallel computations such as image processing. This paper shows a practical 3D VLSI layout of the N-vertex pyramid network with volume $O(N)$ and wire-length $O(\sqrt[3]{N})$. Since the known best lower bounds for the volume and wire-length of a 3D layout for an N-vertex pyramid network are $\Omega(N)$ and $\Omega(\sqrt[3]{N}/\log N)$, respectively, the volume of our layout is optimal, and the wire-length of our layout is close to the optimal.

1. INTRODUCTION

There has been a great interest in embedding graphs into 3D(three-dimensional) grids motivated by 3D VLSI circuits and 3D drawings. This paper is concerned with 3D layouts of pyramid networks which are well-known as suitable structures for parallel computations such as image processing and image understanding [5, 7].

The notion of 3D layout of a VLSI circuit follows the classic 2D framework. The circuit is represented by a graph and the media in which the circuit is to be realized is a 3D rectangular grid. A circuit layout is a vertex-disjoint embedding of the circuit-graph in the grid. The cost of a layout is measured by the volume and wire-length of the layout. It follows from general lower bounds derived by Rosenberg[8] that the volume and wire-length of a 3D layout for an N-vertex pyramid network are $\Omega(N)$ and $\Omega(\sqrt[3]{N}/\log N)$, respectively. On the other hand, it is implicit in [6] by Ng, Pun, Ip, Hamdi, and Ahmad that an N-vertex pyramid network can be laid out in $O(N)$ volume with wire-length at most $O(\sqrt{N})$.

This paper shows direct and simple layouts of pyramid networks. The volume of our layout is optimal, and the wire-length is close to the optimal. More precisely, the volume and wire-length of our layout for an N-vertex pyramid network are $O(N)$ and $O(\sqrt[3]{N})$, respectively.

Since an N-vertex multigrid network is a spanning subgraph of an N-vertex pyramid network, it follows that an N-vertex multigrid network can be laid out in $O(N)$ volume with wire-length at most $O(\sqrt[3]{N})$, which is an improvement on a previous result by Calamoneri and Massini [1].

2. PRELIMINARIES

2.1. Grids

Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. We denote by $\delta_G(v)$ the degree of a vertex $v \in V(G)$ and define $\Delta(G) = \max\{\delta_G(v) : v \in V(G)\}$.

Let $[m] = \{0, 1, \ldots, m-1\}$ for any positive integer m. The d-dimensional $m_1 \times m_2 \times \cdots \times m_d$ grid, denoted by $R(m_1, m_2, \ldots, m_d)$, is the graph defined as follows:

$$V(R(m_1, m_2, \ldots, m_d)) = [m_1] \times [m_2] \times \cdots \times [m_d];$$

$$E(R(m_1, m_2, \ldots, m_d)) = \{(u, v) : \sum_{i=1}^{d} |v_i - u_i| = 1\},$$

where $u = [u_1, u_2, \ldots, u_d]$ and $v = [v_1, v_2, \ldots, v_d]$. (u, v) is called an i-dimensional edge if $|v_i - u_i| = 1$ for some i and $u_j = v_j$ for every $j \neq i$.

A path P on a d-dimensional grid R is called a segment if P consists of i-dimensional edges for some positive integer i. A segment connecting u and v is called (u, v)-segment. Notice that any path on R can be represented as the concatenation of segments. If a path P connecting v_0 and v_k on R is represented as the concatenation of (v_0, v_1)-segment, (v_1, v_2)-segment,..., and (v_{k-1}, v_k)-segment, then we represent P as $(v_0, v_1, v_2, \ldots, v_{k-1}, v_k)$.

Let R be a d-dimensional grid. For any $S \subset V(R)$ and $v \in V(R)$, we define that

$$S + v = \{s + v : s \in S\}.$$

For any path $P = (v_0, v_1, \ldots, v_k)$ on R and $v \in V(R)$,

$$P + v = (v_0 + v, v_1 + v, \ldots, v_k + v).$$

2.2. Pyramid Networks

For any natural number n, the pyramid of height n, denoted by $P(n)$, is the graph defined as follows: $V(P(n)) = \{[x,y,z] : x,y \in [2^{n-z}], z \in [n+1]\}$; Any two vertices $[x,y,z]$ and $[x',y',z']$ are connected by an edge if

- $|x - x'| + |y - y'| = 1$ and $z = z'$, or
- $x' = \lfloor x/2 \rfloor$, $y' = \lfloor y/2 \rfloor$, and $z' = z + 1$.

$([x,y,z],[x',y',z'])$ is called a 1-dimensional edge if $|x - x'| = 1$, $y = y'$, and $z = z'$, a 2-dimensional edge if $x = x'$, $|y - y'| = 1$, and $z = z'$, and a slope-edge otherwise. It is easy to see that $P(n)$ consists of $N = (4^{n+1} - 1)/3$ vertices, and $\Delta(P(n)) = 7$ if $n = 2$ and $\Delta(P(n)) = 9$ if $n \geq 3$.

For any natural numbers n and $k \leq n$, $F_k(n)$ is defined as the subgraph of $P(n)$ induced by $\{[x,y,z] \in V(P(n)) : z \leq k\}$.

For any positive integer m and natural number l, let $[m]_l = \{lm, lm+1, \ldots, (l+1)m - 1\}$. For any natural numbers i, j, n, let $V_{i,j}(n) = \{[x,y,z] : x \in [2^{n-z}]_i, y \in [2^{n-z}]_j, z \in [n+1]\}$, and define $P_{i,j}(n)$ as the following graph: $V(P_{i,j}(n)) = V_{i,j}(n)$; Any two vertices $[x,y,z]$ and $[x',y',z']$ are connected by an edge if

- $|x - x'| + |y - y'| = 1$ and $z = z'$, or
- $x' = \lfloor x/2 \rfloor$, $y' = \lfloor y/2 \rfloor$, and $z' = z + 1$.

For any two graphs G and H, let $G \cup H$ denote the disjoint union of G and H. For any natural numbers n and $k \leq n$, $G_k(n) = \bigcup_{i,j \in [2^{n-k}]} P_{i,j}(k)$. It is easy to see the following.

Theorem 1 *$F_k(n)$ is the graph obtained from $G_k(n)$ by connecting $[(i+1)2^{k-z} - 1, y, z]$ and $[(i+1)2^{k-z}, y, z]$ by an edge for each $i \in [2^{n-k} - 1]$, $y \in [2^{n-z}]$, and $z \in [k+1]$, and connecting $[x, (j+1)2^{k-z} - 1, z]$ and $[x, (j+1)2^{k-z}, z]$ by an edge for each $j \in [2^{n-k} - 1]$, $x \in [2^{n-z}]$, and $z \in [k+1]$.* ∎

For any natural numbers k and n, let $U_k(n) = \{[x, y, z+k] : x, y \in [2^{n-z}], z \in [n+1]\}$, and define $P^k(n)$ as the following graph: $V(P^k(n)) = U_k(n)$; Any two vertices $[x,y,z]$ and $[x',y',z']$ are connected by an edge if

- $|x - x'| + |y - y'| = 1$ and $z = z'$, or
- $x' = \lfloor x/2 \rfloor$, $y' = \lfloor y/2 \rfloor$, and $z' = z + 1$.

2.3. Multigrid Networks

For any natural number n, the multigrid of height n, denoted by $M(n)$, is the graph defined as follows: $V(M(n)) = \{[x,y,z] : x,y \in [2^{n-z}], z \in [n+1]\}$; Any two vertices $[x,y,z]$ and $[x',y',z']$ are connected by an edge if

- $|x - x'| + |y - y'| = 1$ and $z = z'$, or
- $x' = x/2$, $y' = y/2$, and $z' = z + 1$.

It is easy to see that $M(n)$ is a spanning subgraph of $P(n)$ for any n.

2.4. Layouts

An embedding $\langle \phi, \rho \rangle$ of a graph G into a graph H is defined by a one-to-one mapping $\phi : V(G) \to V(H)$, together with a mapping ρ that maps each edge $(u,v) \in E(G)$ onto a path $\rho(u,v)$ in H that connects $\phi(u)$ and $\phi(v)$. The dilation of an embedding $\langle \phi, \rho \rangle$ is the maximum length of $\rho(e)$ over all the edges $e \in E(G)$.

A layout of a graph G into a 3-dimensional grid R is an embedding $\langle \phi, \rho \rangle$ of G into R such that $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint for any distinct $e_1, e_2 \in E(G)$. $\langle \phi, \rho \rangle$ is called a 3-D layout of G if $\langle \phi, \rho \rangle$ is a layout of G into a 3-dimensional grid R, and the dilation of $\langle \phi, \rho \rangle$ is called the wire-length of $\langle \phi, \rho \rangle$. The volume of a 3-D layout $\langle \phi, \rho \rangle$ is the number of vertices in R.

For any integer $n \geq 2$, there exists no 3-D layout of $P(n)$ because $\Delta(P(n)) \geq 7$ and $\Delta(R) \leq 6$ for any 3-dimensional grid R. Thus, we extend the notion of a 3-D layout as follows: ϕ maps $v \in V(G)$ onto a set of two adjacent vertices in R such that $\phi(u) \cap \phi(v) = \emptyset$ for any distinct vertices $v, v' \in V(G)$; For any $e = (u,v) \in E(G)$, $\rho(e)$ is a path in R connecting a vertex in $\phi(u)$ and one in $\phi(v)$ such that $\rho(e)$ and $\rho(e')$ are internally disjoint for any distinct edges $e, e' \in E(G)$.

3. MAIN RESULTS

The purpose of the paper is to prove the following theorem.

Theorem A *$P(n)$ can be laid out in $O(N)$ volume with wire-length at most $O(\sqrt[3]{N})$, where $N = |V(P(n))|$.*

The following is immediate.

Corollary A *$M(n)$ can be laid out in $O(N)$ volume with wire-length at most $O(\sqrt[3]{N})$, where $N = |V(M(n))|$.*

4. PROOF OF THEOREM A

4.1. 3-Layer Layouts

For any natural numbers i, j, and n, we define a mapping $\psi_{i,j,n}$ from $V(P_{i,j}(n))$ to $V(P(n))$ as follows: For any $[x,y,z] \in V(P_{i,j}(n))$,

$$\psi_{i,j,n}([x,y,z]) = [x - i2^{n-z}, y - j2^{n-z}, z].$$

It is easy to see the following.

Lemma 1 *For any natural numbers i, j, and n, $\psi_{i,j,n}$ is an isomorphism from $P_{i,j}(n)$ to $P(n)$.* ∎

For any natural number n, we define an embedding $\langle \phi_n, \rho_n \rangle$ of $P(n)$ into $R(2^{n+2}-3, 2^{n+2}-3, 3)$ as follows.

If $n = 0$ then we define that $\phi_0([0,0,0]) = \{[0,0,0], [0,0,1]\}$ and $\rho_0 : \emptyset \rightarrow E(R(1,1,3))$;

Let n be a positive integer. For any $v \in V(P(n))$, we define $\phi_n(v)$ as follows: If $v \in V(P_{i,j}(n-1))$ for some $i,j \in [2]$ then

$$\phi_n(v) = \phi_{n-1}(\psi_{i,j,n-1}(v)) + [i2^{n+1}, j2^{n+1}, 0],$$

and otherwise, that is if $v = [0,0,n]$,

$$\phi_n(v) = \{[2^{n+1}-2, 2^{n+1}-2, k] : \ k \in [2]\}.$$

We define $\rho_n(e)$ for any $e \in E(P(n))$. If $e = (u,v) \in E(P_{i,j}(n-1))$ for some $i,j \in [2]$ then

$$\rho_n(e) \ = \ \rho_{n-1}((\psi_{i,j,n-1}(u), \psi_{i,j,n-1}(v))) + [i2^{n+1}, j2^{n+1}, 0].$$

Otherwise:

If e is a 1-dimensional edge then $e = ([2^{n-z-1}-1, y, z], [2^{n-z-1}, y, z])$ for some $y \in [2^{n-z}]$ and $z \in [n]$, and $\rho_n(e)$ is defined as $([2^{n+1}-2^{z+1}-2, (2y+1)2^{z+1}-2, 0], [2^{n+1}+2^{z+1}-2, (2y+1)2^{z+1}-2, 0])$-segment;

If e is a 2-dimensional edge then $e = ([x, 2^{n-z-1}-1, z], [x, 2^{n-z-1}, z])$ for some $x \in [2^{n-z}]$ and $z \in [n]$ and $\rho_n(e)$ is defined as $([(2x+1)2^{z+1}-2, 2^{n+1}-2^{z+1}-2, 1], [(2x+1)2^{z+1}-2, 2^{n+1}+2^{z+1}-2, 1])$-segment;

If $e = ([0,0,n-1],[0,0,n])$ then $\rho_n(e)$ is defined as a path $([2^n-2, 2^n-2, 1], [2^n-2, 2^n-2, 2], [2^n-2, 2^{n+1}-3, 2], [2^{n+1}-3, 2^{n+1}-3, 2], [2^{n+1}-3, 2^{n+1}-3, 1], [2^{n+1}-3, 2^{n+1}-2, 1], [2^{n+1}-2, 2^{n+1}-2, 1])$;

If $e = ([0,1,n-1],[0,0,n])$ then $\rho_n(e)$ is defined as a path $([2^n-2, 3\cdot 2^n-2, 1], [2^n-2, 3\cdot 2^n-2, 2], [2^{n+1}-3, 3\cdot 2^n-2, 2], [2^{n+1}-3, 2^{n+1}-1, 2], [2^{n+1}-3, 2^{n+1}-1, 0], [2^{n+1}-2, 2^{n+1}-1, 0], [2^{n+1}-2, 2^{n+1}-2, 0])$;

If $e = ([1,0,n-1],[0,0,n])$ then $\rho_n(e)$ is defined as a path $([3\cdot 2^n-2, 2^n-2, 1], [3\cdot 2^n-2, 2^n-2, 2], [2^{n+1}-1, 2^n-2, 2], [2^{n+1}-1, 2^{n+1}-3, 2], [2^{n+1}-1, 2^{n+1}-3, 0], [2^{n+1}-2, 2^{n+1}-3, 0], [2^{n+1}-2, 2^{n+1}-2, 0])$;

If $e = ([1,1,n-1],[0,0,n])$ then $\rho_n(e)$ is defined as a path $([3\cdot 2^n-2, 3\cdot 2^n-2, 1], [3\cdot 2^n-2, 3\cdot 2^n-2, 2], [3\cdot 2^n-2, 2^{n+1}-1, 2], [2^{n+1}-1, 2^{n+1}-1, 2], [2^{n+1}-1, 2^{n+1}-1, 1], [2^{n+1}-1, 2^{n+1}-2, 1], [2^{n+1}-2, 2^{n+1}-2, 1])$.

Notice that $\langle \phi_n, \rho_n \rangle$ is an embedding of $P_{i,j}(n-1)$, which is isomorphic to $P(n-1)$ by Lemma 1, into the subgrid of $R(2^{n+2}-3, 2^{n+2}-3, 3)$ induced by

$$\{[x,y,z] : x,y \in [2^{n+1}-3], z \in [3]\} + [i2^{n+1}, j2^{n+1}, 0]$$

by using $\langle \phi_{n-1}, \rho_{n-1} \rangle$ for each $i,j \in [2]$.

Lemma 2 *For any $[x,y,z] \in V(P(n))$,*

$$\phi_n([x,y,z]) = \{[(2x+1)2^{z+1}-2, (2y+1)2^{z+1}-2, k] : \ k \in [2]\}.$$

Proof : The lemma is proved by induction on n. The lemma holds for $n = 0$ since $\phi_0([0,0,0]) = \{[0,0,0], [0,0,1]\}$.

Let n be a positive integer. Assume for inductive step that for any $[x',y',z'] \in V(P(n-1))$,

$$\phi_{n-1}([x',y',z']) = \{[(2x'+1)2^{z'+1}-2, (2y'+1)2^{z'+1}-2, k] : k \in [2]\}.$$

Notice that for any integers x,y,z,i,j,

$$\{2(x - i2^{n-z-1})+1\}2^{z+1} - 2 + i2^{n+1} = (2x+1)2^{z+1}-2 \text{ and}$$
$$\{2(y - j2^{n-z-1})+1\}2^{z+1} - 2 + j2^{n+1} = (2y+1)2^{z+1}-2.$$

If $[x,y,z] \in V(P_{i,j}(n-1))$ for some $i,j \in [2]$ then

$$\psi_{i,j,n-1}([x,y,z]) = [x - i2^{n-z-1}, y - j2^{n-z-1}, z],$$

and therefore, we obtain by the induction hypothesis

$$\begin{aligned}\phi_n([x,y,z]) &= \phi_{n-1}(\psi_{i,j,n-1}([x,y,z])) + [i2^{z+1}, j2^{z+1}, 0] \\ &= \{[(2x+1)2^{z+1}-2, (2y+1)2^{z+1}-2, k] : k \in [2]\}.\end{aligned}$$

Since

$$\phi_n([0,0,n]) = \{[2^{n+1}-2, 2^{n+1}-2, k] : k \in [2]\},$$

we conclude that

$$\phi_n([x,y,z]) = \{[(2x+1)2^{z+1}-2, (2y+1)2^{z+1}-2, k] : k \in [2]\}$$

for any $[x,y,z] \in V(P(n))$. ∎

It is easy to see the following.

Lemma 3 *For any four natural numbers x, z, x', and z', $(2x+1)2^{z+1} = (2x'+1)2^{z'+1}$ if and only if $[x,z] = [x',z']$.* ∎

Theorem 2 *$\langle\phi_n,\rho_n\rangle$ is a layout of $P(n)$ into $R(2^{n+2}-3, 2^{n+2}-3, 3)$ with wire-length at most $2^{n+1}+3$.*

Proof : The theorem is proved by induction on n. Trivially, $\langle\phi_0,\rho_0\rangle$ is a layout of $P(0)$ into $R(1,1,3)$ with wire-length $0 \le 2^1+3=5$.

Assume that $\langle\phi_{n-1},\rho_{n-1}\rangle$ is a layout of $P(n-1)$ into $R(2^{n+1}-3, 2^{n+1}-3, 3)$ with wire-length at most 2^n+3. By Lemmas 2 and 3, $\phi_n(v)\cap\phi_n(v')=\emptyset$ for any distinct $v,v' \in V(P(n))$. By the definition of $\langle\phi_n,\rho_n\rangle$, it is easy to see that $\rho_n(e)$ is a path connecting a vertex in $\phi_n(u)$ and that in $\phi_n(v)$ for any $e=(u,v)\in E(P(n))$. Let

$$\begin{aligned}
E_0 &= E(G_{n-1}(n)) = \bigcup_{i,j\in[2]} E(P_{i,j}(n-1))\\
E_1 &= \{([2^{n-z-1}-1, y, z], [2^{n-z-1}, y, z]) : \\
&\qquad y \in [2^{n-z}], z\in[n]\}\\
E_2 &= \{([x, 2^{n-z-1}-1, z], [x, 2^{n-z-1}, z]) : \\
&\qquad x \in [2^{n-z}], z\in[n]\}\\
E_3 &= \{([x,y,n-1],[0,0,n]) :\ x,y\in[2]\}.
\end{aligned}$$

It is easy to see that (E_0,E_1,E_2,E_3) is a partition of $E(P(n))$. Since $\langle\phi_{n-1},\rho_{n-1}\rangle$ is a layout of $P(n-1)$ into $R(2^{n+1}-3, 2^{n+1}-3, 3)$ by the induction hypothesis, $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint for any distinct $e,e'\in E_0$. It is not difficult to prove that for any $e\in E_0$ and $e'\in E(P(n))-E_0$, $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint. For any distinct $e,e'\in E_i$ ($i=1$ or 2), $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint by Lemma 3. For any distinct $e,e'\in E_3$, $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint by definition. It is easy to see that for any $e\in E_i$ and $e'\in E_j$ with $i\ne j$ ($i,j\in\{1,2,3\}$), $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint. Thus, we conclude that $\rho_n(e)$ and $\rho_n(e')$ are internally disjoint for any distinct edges $e,e'\in E(P(n))$. Hence, $\langle\phi_n,\rho_n\rangle$ is a layout of $P(n)$ into $R(2^{n+2}-3, 2^{n+2}-3, 3)$. It is easy to see that the wire-length of $\langle\phi_n,\rho_n\rangle$ is at most $2^{n+1}+3$. ∎

4.2. 3-D Layouts

We denote by $Q_k(n)$ the graph obtained from $F_k(n)\cup P^{k+1}(n-k)$ by connecting $[x,y,k]$ and $[x,y,k+1]$ by an edge for each $x,y\in[2^{n-k}]$.

Lemma 4 *For any natural numbers n and $k\le n$, there exists an embedding $\langle\phi',\rho'\rangle$ of $P(n)$ into $Q_k(n)$ with dilation 2 such that $\rho'(e)$ and $\rho'(e')$ are internally disjoint for any distinct edges $e,e'\in E(P(n))$.*

Proof : For any $v=[x,y,z]\in V(P(n))$, we define that

$$\phi'(v) = \begin{cases} [x,y,z] & \text{if } z\le k,\\ [x,y,z+1] & \text{otherwise.}\end{cases}$$

For any $e=([x,y,z],[x',y',z'])\in E(P(n))$, we define $\rho'(e)$ as an edge $([x,y,z],[x',y',z'])$ if $z,z'\le k$, an edge $([x,y,z+1],[x',y',z'+1])$ if $z,z'\ge k+1$, and the path consisting of two edges $([x,y,k],[x,y,k+1])$ and $([x,y,k+1],[x',y',k+2])$ if $x'=\lceil x/2\rceil$, $y'=\lceil y/2\rceil$, $z=k$, and $z'=k+1$. It is easy to see that $\langle\phi',\rho'\rangle$ is a desired embedding. ∎

In this subsection, we show a layout $\langle\varphi_n,\varrho_n\rangle$ of $Q_m(n)$ into $R(2^{m+2}+2^{l+1}-3, 2^{m+2}+4, 3\cdot 2^{2l})$, where $m=\lceil 2n/3\rceil$ and $l=n-m$. $\langle\varphi_n,\varrho_n\rangle$ naturally induces a layout of $P(n)$ into $R(2^{m+2}+2^{l+1}-3, 2^{m+2}+4, 3\cdot 2^{2l})$ together with $\langle\phi',\rho'\rangle$ in Lemma 4.

4.2.1. 3-D Layouts of $F_m(n)$

For any natural numbers i and n, we define a mapping $\sigma_{i,n}$ from $[2^n]_i$ to $[2^n]$ as follows: For any $x\in[2^n]_i$,

$$\sigma_{i,n}(x) = \begin{cases} x-i2^n & \text{if } i \text{ is even},\\ (i+1)2^n-1-x & \text{otherwise.}\end{cases}$$

Notice that $\sigma_{i,n}$ is a bijection, and

$$\sigma^{-1}_{i,n}(x') = \begin{cases} x'+i2^n & \text{if } i \text{ is even},\\ (i+1)2^n-1-x' & \text{otherwise.}\end{cases}$$

Lemma 5 *Let n be a natural number. If $\sigma^{-1}_{i,n}(x') = \sigma^{-1}_{j,n}(y')$ for some natural numbers i, j, and $x',y'\in[2^n]$ then $[i,x']=[j,y']$.*

Proof : The lemma follows from the fact that $i=\lfloor\sigma^{-1}_{i,n}(x')/2^n\rfloor$. ∎

We define a mapping $\eta_{i,j,n}$ from $V(P_{i,j}(n))$ to $V(P(n))$ as follows:

$$\mu_{i,j,n}([x,y,z]) = [\sigma_{i,n-z}(x), \sigma_{j,n-z}(y), z].$$

It is easy to see the following.

Lemma 6 *$\mu_{i,j,n}$ is an isomorphism from $P_{i,j}(n)$ to $P(n)$.* ∎

We define an embedding $\langle\varphi'_n,\varrho'_n\rangle$ of $F_m(n)$ into $R(2^{m+2}+2^{l+1}-3, 2^{m+2}-1, 3\cdot 2^{2l})$ as follows.

If $v\in V(P_{i,j}(m))$ for some $i,j\in[2^l]$ then

$$\varphi'_n(v) = \phi_m(\mu_{i,j,m}(v)) + [2^l, 1, 3\sigma^{-1}_{i,l}(j)].$$

If $e=(u,v)\in E(P_{i,j}(m))$ for some $i,j\in[2^l]$ then

$$\begin{aligned}\varrho'_n(e) &= \rho_m((\mu_{i,j,m}(u),\mu_{i,j,m}(v)))\\ &\qquad + [2^l, 1, 3\sigma^{-1}_{i,l}(j)].\end{aligned}$$

For any $e\in E(F_m(n))-E(G_m(n))$, we define $\varrho'_n(e)$ as follows.

(Case 1) e is a 1-dimensional edge: $e = ([(i+1)2^{m-z}-1, j2^{m-z}+\beta, z], [(i+1)2^{m-z}, j2^{m-z}+\beta, z])$ for some $i \in [2^l - 1]$, $j \in [2^l]$, $z \in [m+1]$, and $\beta \in [2^{m-z}]$. There are four subcases.

(Case 1.1) i and j are even: $\varrho'_n(e)$ is defined as a path $([2^{m+2} + 2^l - 2^{z+1} - 2, (2\beta+1)2^{z+1} - 1, 3(i2^l + j)], [2^{m+2}+2^{l+1}-4-j, (2\beta+1)2^{z+1}-1, 3(i2^l+j)], [2^{m+2}+2^{l+1}-4-j, (2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^{m+2}+2^l+2^{z+1}-2, (2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)])$.

(Case 1.2) i is even and j is odd: $\varrho'_n(e)$ is defined as a path $([2^{m+2}+2^l-2^{z+1}-2, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3(i2^l+j)], [2^{m+2}+2^{l+1}-4-j, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3(i2^l+j)], [2^{m+2}+2^{l+1}-4-j, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^{m+2}+2^l-2^{z+1}-2, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)])$.

(Case 1.3) i is odd and j is even: $\varrho'_n(e)$ is defined as a path $([2^l+2^{z+1}-2, (2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^l-j-1, (2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^l-j-1, (2\beta+1)2^{z+1}-1, 3((i+1)2^l+j)], [2^l+2^{z+1}-2, (2\beta+1)2^{z+1}-1, 3((i+1)2^l+j)])$.

(Case 1.4) i and j are odd: $\varrho'_n(e)$ is defined as a path $([2^l+2^{z+1}-2, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^l-j-1, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l-j-1)], [2^l-j-1, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l+j)], [2^l+2^{z+1}-2, 2^{m+2}-(2\beta+1)2^{z+1}-1, 3((i+1)2^l+j)])$.

(Case 2) e is a 2-dimensional edge: $e = ([i2^{m-z}+\alpha, (j+1)2^{m-z}-1, z], [i2^{m-z}+\alpha, (j+1)2^{m-z}, z])$ for some $i \in [2^l]$, $j \in [2^l-1]$, $z \in [m+1]$, and $\alpha \in [2^{m-z}]$. There are five subcases.

(Case 2.1) i and j are even, and $z \neq m$: $\varrho'_n(e)$ is defined as a path $([2^l+(2\alpha+1)2^{z+1}-2, 2^{m+2}-2^{z+1}-1, 3(i2^l+j)+1], [2^l+(2\alpha+1)2^{z+1}-2, 2^{m+2}-2, 3(i2^l+j)+1], [2^l+(2\alpha+1)2^{z+1}-2, 2^{m+2}-2, 3(i2^l+j+1)+1], [2^l+(2\alpha+1)2^{z+1}-2, 2^{m+2}-2^{z+1}-1, 3(i2^l+j+1)+1])$.

(Case 2.2) i is even and j is odd, and $z \neq m$: $\varrho'_n(e)$ is defined as a path $([2^l+(2\alpha+1)2^{z+1}-2, 2^{z+1}-1, 3(i2^l+j)+1], [2^l+(2\alpha+1)2^{z+1}-2, 0, 3(i2^l+j)+1], [2^l+(2\alpha+1)2^{z+1}-2, 0, 3(i2^l+j+1)+1], [2^l+(2\alpha+1)2^{z+1}-2, 2^{z+1}-1, 3(i2^l+j+1)+1])$.

(Case 2.3) i is odd and j is even, and $z \neq m$: $\varrho'_n(e)$ is defined as a path $([2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{m+2}-2^{z+1}-1, 3((i+1)2^l-j-1)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{m+2}-2, 3((i+1)2^l-j-1)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{m+2}-2, 3((i+1)2^l-j-2)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{m+2}-2^{z+1}-1, 3((i+1)2^l-j-2)+1])$.

(Case 2.4) i and j are odd, and $z \neq m$: $\varrho'_n(e)$ is defined as a path $([2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{z+1}-1, 3((i+1)2^l-j-1)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 0, 3((i+1)2^l-j-1)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 0, 3((i+1)2^l-j-2)+1], [2^{m+2}+2^l-(2\alpha+1)2^{z+1}-2, 2^{z+1}-1, 3((i+1)2^l-j-2)+1])$.

(Case 2.5) $z = m$, that is $u = [i, j, m]$ and $v = [i, j+1, m]$: If i is even then $\varrho'_n(e)$ is defined as $([2^{m+1}+2^l-2, 2^{m+1}-1, 3(i2^l+j)+1], [2^{m+1}+2^l-2, 2^{m+1}-1, 3(i2^l+j+1)])$-segment. If i is odd then $\varrho'_n(e)$ is defined as $([2^{m+1}+2^l-2, 2^{m+1}-1, 3((i+1)2^l-j-1)], [2^{m+1}+2^l-2, 2^{m+1}-1, 3((i+1)2^l-j-2)+1])$-segment.

It should be noted that $\langle \varphi'_n, \varrho'_n \rangle$ is an embedding of $P_{i,j}(m)$ into the subgrid of $R(2^{m+2}+2^{l+1}-3, 2^{m+2}-1, 3\cdot 2^{2l})$ induced by $\{[x+2^l, y+1, z+3\sigma_{i,l}^{-1}(j)] : x, y \in [2^{m+2}-3], z \in [3]\}$ by using $\mu_{i,j,m}$ and $\langle \phi_m, \rho_m \rangle$ for each $i, j \in [2^l]$.

Theorem 3 *$\langle \varphi'_n, \varrho'_n \rangle$ is a layout of $F_m(n)$ into $R(2^{m+2}+2^{l+1}-3, 2^{m+2}-1, 3\cdot 2^{2l})$ with wire-length at most $2^{m+2}+2^{l+3}-7$, where $m = \lceil 2n/3 \rceil$ and $l = n-m$.*

Proof : By Theorem 2 and Lemmas 5 and 6, $\varphi'_n(v) \cap \varphi'_n(v') = \emptyset$ for any distinct $v, v' \in V(F_m(n))$. It is easy to see that $\rho_n(e)$ is a path connecting a vertex in $\varphi'_n(u)$ and one in $\varphi'_n(v)$ for any $e = (u,v) \in E(F_m(n))$. It is not difficult to prove that $\varrho'_n(e)$ and $\varrho'_n(e')$ are internally disjoint for any distinct edges $e, e' \in E(F_m(n))$, and the wire-length of $\langle \varphi'_n, \varrho'_n \rangle$ is at most $2^{m+2}+2^{l+3}-7$. ∎

4.2.2. 3-D Layouts of $Q_m(n)$

For any natural numbers k and n, we define a mapping $\tau_{k,n}$ from $V(P^k(n))$ to $V(P(n))$ as follows: For any $[x,y,z] \in V(P^k(n))$,

$$\tau_{k,n}([x,y,z]) = [x, y, z-k].$$

It is easy to see the following.

Lemma 7 *For any natural numbers k and n, $\tau_{k,n}$ is an isomorphism of $P^k(n)$ into $P(n)$.* ∎

We define an embedding $\langle \varphi_n, \varrho_n \rangle$ of $Q_m(n)$ into $R = R(2^{m+2}+2^{l+1}-3, 2^{m+2}+4, 3\cdot 2^{2l})$ as follows. (See Figure 1.)

(Case 1) Subgraph $F_m(n)$: Embeds $F_m(n)$ into the subgrid of R induced by $\{[x,y,z] \in V(R) : y \in [2^{m+2}-1]\}$ by using $\langle \varphi'_n, \varrho'_n \rangle$;

(Case 2) Subgraph $P^{m+1}(l)$: Embeds $P^{m+1}(l)$ into the subgrid of R induced by $\{[x,y,z] \in V(R) : y \geq 2^{m+2}+1\}$ as follows: Layout $P^{m+1}(l)$ into $R(2^{l+2}-3, 2^{l+2}-3, 3)$ by using $\tau_{m+1,l}$ and $\langle \phi_l, \rho_l \rangle$; Layout $R(2^{l+2}-3, 2^{l+2}-3, 3)$ into $R(2^{l+2}-3, 3, 2^{l+2}-3)$ by mapping $[x,y,z]$ into $[y,z,x]$; Layout $R(2^{l+2}-3, 3, 2^{l+2}-3)$ into $R(2^{l+2}-3, 3, 3\cdot 2^l(2^l-1)+1)$ by mapping $[x,y,z]$ into $[x, y, 3\cdot 2^{l-2}z]$; Layout $R(2^{l+2}-3, 3, 3\cdot 2^l(2^l-1)+1)$ into R by mapping $[x,y,z]$ to $[x, y+2^{m+2}+1, z]$. Then,

$$\begin{aligned}\varphi_n([i,j,m+1]) \\ = \{[4j, 2^{m+2}+1+k, (3i)2^l] : k \in [2]\}\end{aligned}$$

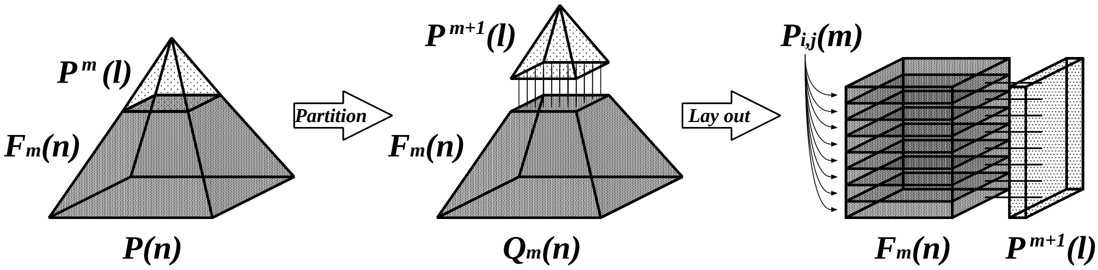

Figure 1: Layout of the Pyramid $P(n)$

for each $i, j \in [2^l]$.

(Case 3) Edge $e = ([i, j, m], [i, j, m+1])$ for each $i, j \in [2^l]$: we define $\varrho_n(e)$ as $([2^{m+1} + 2^l - 2, 2^{m+1} - 1, 3\sigma_{i,l}^{-1}(j)+1], [2^{m+1}+2^l-2, 2^{m+2}-1, 3\sigma_{i,l}^{-1}(j)+1], [4j, 2^{m+2}-1, 3\sigma_{i,l}^{-1}(j)+1], [4j, 2^{m+2}, 3\sigma_{i,l}^{-1}(j)+1], [4j, 2^{m+2}, (3i)2^l], [4j, 2^{m+2}+1, (3i)2^l])$.

Theorem 4 *$\langle \varphi_n, \varrho_n \rangle$ is a layout of $Q_m(n)$ into $R(2^{m+2} + 2^{l+1} - 3, 2^{m+2}+4, 3\cdot 2^{2l})$ with wire-length at most $2^{m+2} + 2^{l+3} - 7$.*

Proof : First, we prove that $\varphi_n(v) \cap \varphi_n(v') = \emptyset$ for any distinct vertices $v, v' \in V(Q_m(n))$. By Theorem 3, $\varphi_n(v) \cap \varphi_n(v') = \emptyset$ for any distinct vertices $v, v' \in V(F_m(n))$. By Theorem 2 and Lemma 7, $\varphi_n(v) \cap \varphi_n(v') = \emptyset$ for any distinct vertices $v, v' \in V(P^{m+1}(l))$. Since

$$\max\{y : [x,y,z] \in \varphi_n(v), v \in V(F_m(n))\} \leq 2^{m+2} - 3 \text{ and}$$
$$\min\{y : [x,y,z] \in \varphi_n(v), v \in V(P^{m+1}(l))\} \geq 2^{m+2},$$

$\varphi_n(v) \cap \varphi_n(v') = \emptyset$ for any $v \in V(F_m(n))$ and $v' \in V(P^{m+1}(l))$. Hence, $\varphi_n(v) \cap \varphi_n(v') = \emptyset$ for any distinct vertices $v, v' \in V(Q_m(n))$.

Next, we prove that $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any distinct edges $e, e' \in E(Q_m(n))$. By Theorem 3, $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any distinct edges $e, e' \in E(F_m(n))$. By Theorem 2, $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any distinct edges $e, e' \in E(P^{m+1}(l))$. It is easy to see that $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any $e \in E(P^{m+1}(l))$ and $e' \in E(Q_m(n)) - E(P^{m+1}(l))$. It is not difficult to see that $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any $e \in E(F_m(n))$ and $e' = ([i, j, m], [i, j, m+1])$ $(i, j \in [2^m])$. Hence, $\varrho_n(e)$ and $\varrho_n(e')$ are internally disjoint for any distinct edges $e, e' \in E(Q_m(n))$.

It is easy to see that the wire-length of $\langle \varphi_n, \varrho_n \rangle$ is at most $2^{m+2} + 2^{l+3} - 7$ by Theorem 3. ∎

Theorem 4 and Lemma 4 complete the proof of Theorem A. ∎

Acknowledgments

The research is a part of CAD21 Project at TIT.

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