LETTER

# On-Line Multicasting in All-Optical Networks

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**SUMMARY** We consider the routing for a multicast in a WDM all-optical network. We prove a min-max theorem on the number of wavelengths necessary for routing a multicast. Based on the min-max theorem, we propose an efficient on-line algorithm for routing a multicast.

***key words:*** *all-optical network, wavelength division multiplexing, multicast, on-line algorithm*

## 1. Introduction

A WDM (Wavalength Division Multiplexing) all-optical network consists of routing nodes interconnected by point-to-point unidirectional fiber-optic links, which support a certain number of wavelengths. The same wavelength on two input ports cannot be routed to the same output port due to the interference. An optical routing assigns a path and a wavelength for each communication request in such a way that no two paths traversing a common link are assigned the same wavelength. A fundamental problem for WDM all-optical networks is to minimize the number of wavelengths necessary for the optical routing. This paper considers the on-line optical routing for a special collection of communication requests called a multicast.

A WDM all-optical network is modeled as a symmetric digraph (directed graph) $G$ with vertex set $V(G)$ and arc (directed edge) set $A(G)$ such that if $(u,v) \in A(G)$ then $(v,u) \in A(G)$, where the vertices represent the routing nodes and each arc represents a point-to-point unidirectional fiber-optic link connecting a pair of routing nodes.

Let $P(x,y)$ denote a dipath (directed path) in $G$ from the vertex $x$ to $y$ which consistis of consecutive arcs beginning at $x$ and ending at $y$. A request is an ordered pair of vertices $(x,y)$ in $G$ corresponding to a message to be sent from $x$ to $y$, and an instance $I$ is a collection (multiset) of requests. A routing for an instance $I$ is a collection of dipaths $R = \{P(x,y)|(x,y) \in I\}$.

Given a symmetric digraph $G$, an instance $I$, and a routing $R$ for $I$, $\omega(G,I,R)$ is the minimum number of wavelengths that can be assigned to the dipaths in $R$, so that no two dipaths sharing an arc have the same wavelength. Let $\omega(G,I)$ denote the smallest $\omega(G,I,R)$ over all routings $R$ for $I$. The load of an arc $\alpha \in A(G)$ in $R$, denoted by $\pi(G,I,R,\alpha)$, is the number of dipaths in $R$ containing $\alpha$. Let $\pi(G,I,R)$ denote the largest $\pi(G,I,R,\alpha)$ over all arcs $\alpha \in A(G)$, and $\pi(G,I)$ denote the smallest $\pi(G,I,R)$ over all routings $R$ for $I$. It is known that computing $\omega(G,I)$ and $\pi(G,I)$ is NP-hard in general [2]. It is not difficult to see that $\omega(G,I) \geqq \pi(G,I)$ for an instance $I$ in a symmetric digraph $G$ and that the inequality can be strict in general [2].

Beauquier, Hell, and Perennes [3] proved that for a multicast $I$ in a symmetric digraph $G$, $\omega(G,I) = \pi(G,I)$ and both $\omega(G,I)$ and $\pi(G,I)$ can be computed in polynominal time. An instance $I$ is called a multicast if $I$ is of the form $\{(x,y)|y \in Y\}$ for a fixed vertex $x \in V(G)$, called the source, and a collection $Y$ of vertices in $V(G)$, called the destinations.

This paper shows a min-max equality on $\omega(G,I)$ for a multicast $I$ in a symmetric digraph $G$ by means of the cut in $G$. For a digraph $G$ and a nonempty proper subset $S \subset V(G)$, a cut $(S,\overline{S})$ is the set of arcs beginning in $S$ and ending in $\overline{S}$, where $\overline{S} = V(G) - S$. For a multicast $I = \{(x,y)|y \in Y\}$ and a cut $(X,\overline{X})$ with $x \in X \subset V(G)$, let $\mu(G,I,X)$ denote $\lceil |Y \cap \overline{X}|/|(X,\overline{X})| \rceil$, and $\mu(G,I)$ denote the largest $\mu(G,I,X)$ over all cuts $(X,\overline{X})$ with $x \in X \subset V(G)$. Notice that $\mu(G,I,X)$ is a lower bound on the average load of an arc in $(X,\overline{X})$ for any routing for $I$. We prove a min-max equality that $\omega(G,I) = \mu(G,I)$, which is used as a basis for on-line multicasting. Let $\delta(x)$ denote the outdegree of $x$ and $\lambda(x)$ denote $\min\{|(X,\overline{X})||x \in X \subset V(G)\}$. Notice that $\delta(x) \geqq \lambda(x)$. If $I$ is a broadcast, that is $I = \{(x,y)|y \in V(G) - x\}$ and $\delta(x) = \lambda(x)$ then our min-max equality implies that $\omega(G,I) = \lceil (|V(G)| - 1)/\delta(x) \rceil$, which is essentially Theorem 3.1 in [4] proved by Bermond, Gargano, Perennes, Rescigno, and Vaccaro.

Given a symmetric digraph $G$ and a sequence of requests $(x_i, y_i)$, an on-line algorithm assigns a dipath $P(x_i, y_i)$ and a wavelength to $P(x_i, y_i)$, so that no two dipaths sharing an arc are assigned the same wavelength. The performance measure for an on-line algorithm is the competitive ratio defined as the worst-case ratio over all request sequences between the number of wavelengths used by the on-line algorithm and the optimal number of wavelengths necessary on the same

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sequence. Bartal and Leonardi [1] showed on-line algorithms with competitive ratio of $O(\log N)$ for any instances in $N$-vertex digraphs associated with meshs, trees, and trees of rings, where the digraph associated with a graph $H$ is the symmetric digraph obtained when each edge $e$ of $H$ is replaced by two oppositely oriented arcs with the same ends as $e$. They also proved a matching lower bound of $\Omega(\log N)$ for digraphs associated with meshes, and a lower bound of $\Omega(\log N/\log\log N)$ for digraphs associated with trees and trees of rings [1].

We show here an on-line algorithm for a multicast $I = \{(x, y)|y \in Y\}$ in a symmetric digraph $G$. We prove that the competitive ratio of our algorithm is $\lceil\delta(x)/\lambda(x)\rceil$. It follows that if $\delta(x) = O(1)$ then the competitive ratio of our algorithm is $O(1)$. Moreover, if $\delta(x) = \lambda(x)$ then our algorithm is optimal. We also show a complementary result that if $\delta(x) > \lambda(x)$ then there is no optimal on-line algorithm. Moreover, we show that the competitive ratio of any on-line algorithm is at least 4/3. We also consider the dynamic multicasting.

## 2. Off-Line Multicasting

We prove in this section the following min-max equality, which will be used in the subsequent sections.

**Theorem 1:** $\omega(G, I) = \mu(G, I)$ for a multicast $I$ in a symmetric digraph $G$.

### 2.1 Proof of Theorem 1

Let $G$ be a symmetric digraph and $I = \{(x, y)|y \in Y\}$ be a multicast in $G$.

#### 2.1.1 Proof of $\omega(G, I) \geqq \mu(G, I)$

It is well-known and easily verified that

$$\omega(G, I) \geqq \pi(G, I). \tag{1}$$

Since $\mu(G, I, X)$ is a lower bound on the average load of an arc in a cut $(X, \overline{X})$ with $x \in X \subset V(G)$ for any routing $R$ for $I$, we have

$$\pi(G, I, R) \geqq \mu(G, I, X)$$

for any routing $R$ for $I$ and any cut $(X, \overline{X})$ with $x \in X \subset V(G)$. Thus, it follows that

$$\pi(G, I) \geqq \mu(G, I). \tag{2}$$

Combining (1) and (2), we have

$$\omega(G, I) \geqq \mu(G, I).$$

#### 2.1.2 Proof of $\omega(G, I) \leqq \mu(G, I)$

It is proved in [3] that for a multicast $I = \{(x, y)|y \in Y\}$ in a symmetric digraph $G$ we have

$$\omega(G, I) = \pi(G, I), \tag{3}$$

by using flow networks derived from $G$.

In a flow network, we denote by $c(u, v)$ the capacity of an arc $(u, v)$, and by $c(T, \overline{T})$ the capacity of a cut $(T, \overline{T})$. Although $Y$ is a collection (multiset) in general, we assume without loss of generality that $Y$ is just a set, as mentioned in [3].

In order to compute $\pi(G, I)$ the following flow network $F_p$ is introduced in [3]. Let $s$ and $t$ be two new vertices which will be the source and sink in $F_p$, respectively. The flow network $F_p$ is defined as follows:

$$V(F_p) = \{s, t\} \cup V(G)$$
$$A(F_p) = \{(s, x)\} \cup A(G) \cup \Big(\bigcup_{y \in Y} \{(y, t)\}\Big)$$
$$c(s, x) = \infty$$
$$c(u, v) = p \text{ for all } (u, v) \in A(G)$$
$$c(y, t) = 1 \text{ for all } y \in Y.$$

The theorem below directly follows from the definition.

**Theorem I:** [3] $\pi(G, I) \leqq p$ if and only if $F_p$ has a flow of value $|Y|$.

By (3) and Theorem I above, it suffices to show that $F_{\mu(G,I)}$ has a flow of value $|Y|$. We prove this by showing that any cut in $F_{\mu(G,I)}$ separating $s$ and $t$ has capacity at least $|Y|$. Any cut in $F_{\mu(G,I)}$ separating $s$ and $t$ can be represented as $(S \cup \{s\}, \overline{S} \cup \{t\})$ for a subset $S$ of $V(G)$ and $\overline{S} = V(G) - S$. It is easy to see that

$$c(S \cup \{s\}, \overline{S} \cup \{t\}) = \begin{cases} |Y \cap S| + \mu(G, I) \cdot |(S, \overline{S})| & \text{if } x \in S \\ \infty & \text{if } x \in \overline{S} \end{cases}$$

where $(S, \overline{S})$ is a cut in $G$. It follows that we may assume that $x \in S$. Then we have

$$\begin{aligned} c(S \cup \{s\}, \overline{S} \cup \{t\}) &= |Y \cap S| + \mu(G, I) \cdot |(S, \overline{S})| \\ &= |Y \cap S| \\ &\quad + \max\left\{ \left\lceil \frac{|Y \cap \overline{X}|}{|(X, \overline{X})|} \right\rceil \,\middle|\, x \in X \subset V(G) \right\} \cdot |(S, \overline{S})| \\ &\geqq |Y \cap S| + \left\lceil \frac{|Y \cap \overline{S}|}{|(S, \overline{S})|} \right\rceil \cdot |(S, \overline{S})| \\ &\geqq |Y \cap S| + \frac{|Y \cap \overline{S}|}{|(S, \overline{S})|} \cdot |(S, \overline{S})| \\ &= |Y \cap S| + |Y \cap \overline{S}| = |Y|, \end{aligned}$$

as desired.

## 3. On-Line Multicasting

### 3.1 Upper Bounds

Let $G$ be a symmetric digraph, and $(x, y_1), (x, y_2), \cdots,$

$(x, y_j), \cdots$ be a sequence of multicast requests in $G$. Let $I_j$ denote the collection $\{(x, y_1), (x, y_2), \cdots, (x, y_j)\}$, and $Y_j$ denote the collection $\{y_1, y_2, \cdots, y_j\}$. We assume without loss of generality that $x$ is not a cut-vertex in $G$. We also assume that the wavelengths are labeled with positive integers. Our on-line algorithm is based on a classic theorem of Edmonds [5]. For a vertex $u$ of a digraph $G$, $u$-arborescence $H(u)$ in $G$ is an acyclic spanning subdigraph of $G$ such that for every vertex $v \in V(G)$ there is exactly one dipath in $H(u)$ from $u$ to $v$.

**Theorem II:** [5] For a digraph $G$ and a vertex $u \in V(G)$, the maximum number of arc-disjoint $u$-arborescences in $G$ is equal to $\lambda(u)$.

Let $\mathcal{H} = \{H_1(x), H_2(x), \cdots, H_{\lambda(x)}(x)\}$ be a set of arc-disjoint $x$-arborescences in $G$. For each request, our on-line algorithm, called ARB, assigns a dipath in an $x$-arborescence in $\mathcal{H}$. Given a request $(x, y_j)$, ARB finds an $x$-arborescence $H_k(x)$ such that the number of dipaths in $H_k(x)$ assigned to the existing requests is minimal, assigns the unique dipath $P(x, y_j)$ in $H_k(x)$, and assigns the lowest available wavelength to $P(x, y_j)$.

**Theorem 2:** The competitive ratio of ARB is $\lceil \delta(x)/\lambda(x) \rceil$.

**Proof:** From Theorem 1, we have that for any $j$,

$$\begin{aligned} \omega(G, I_j) &= \mu(G, I_j) \\ &= \max\left\{ \left\lceil \frac{|Y_j \cap \overline{X}|}{|(X, \overline{X})|} \right\rceil \;\middle|\; x \in X \subset V(G) \right\} \\ &\geqq \left\lceil \frac{|Y_j \cap (V(G) - \{x\})|}{|(\{x\}, V(G) - \{x\})|} \right\rceil \\ &= \left\lceil \frac{|Y_j|}{\delta(x)} \right\rceil \\ &\geqq \frac{|Y_j|}{\delta(x)}. \end{aligned}$$

Let $\omega(G, I_j, \mathrm{ALG})$ denote the number of wavelengths used by an on-line algorithm ALG for $I_j$. We have that

$$\begin{aligned} \omega(G, I_j, \mathrm{ARB}) &= \left\lceil \frac{|Y_j|}{\lambda(x)} \right\rceil \\ &\leqq \left\lceil \frac{\omega(G, I_j) \cdot \delta(x)}{\lambda(x)} \right\rceil \\ &\leqq \left\lceil \frac{\delta(x)}{\lambda(x)} \right\rceil \cdot \omega(G, I_j), \end{aligned}$$

as desired. □

The following corollaries are immediate. An on-line algorithm ALG is said to be optimal for $G$ if $\omega(G, I_j, \mathrm{ALG}) = \omega(G, I_j)$ for any $j$.

**Corollary 1:** If $\delta(x)$ is $O(1)$ then the competitive ratio of ARB is $O(1)$.

**Corollary 2:** If $\delta(x) = \lambda(x)$ then ARB is optimal for $G$.

**Corollary 3:** ARB is optimal for digraphs associated with trees, cycles, tori, hypercubes, and cube-connected cycles.

### 3.2 Lower Bounds

The following is a complementary result to Corollary 2.

**Theorem 3:** If $\delta(x) > \lambda(x)$ then there is no on-line algorithm optimal for $G$.

**Proof:** We prove the theorem by contradiction. Let $G$ be a symmetric digraph, and $x$ be a vertex in $G$ with $\delta(x) > \lambda(x)$. Assume that there is an on-line algorithm ALG optimal for $G$. Let $(X, \overline{X})$ be a cut in $G$ such that $x \in X \subset V(G)$ and $|(X, \overline{X})| = \lambda(x)$, and $v$ be a vertex in $\overline{X}$. We denote the arcs with tail $x$ by $(x, u_1), (x, u_2), \cdots, (x, u_{\delta(x)})$. We consider the following sequence of requests:

$$(x, u_1), (x, u_2), \cdots, (x, u_{\delta(x)}), \underbrace{(x, v), (x, v), \cdots, (x, v)}_{\lambda(x)+1}.$$

Since ALG is optimal for $G$, ALG assigns for the requests $(x, u_i)$ arc-disjoint dipaths $P(x, u_i)$ and the same wavelength, say $w$, to the dipaths $P(x, u_i)$ $(1 \leqq i \leqq \delta(x))$. Notice that each arc $(x, u_i)$ is contained in the dipaths assigned wavelength $w$ $(1 \leqq i \leqq \delta(x))$. Since $|(X, \overline{X})| = \lambda(x)$, ALG uses at least two more wavelengths different from $w$ for the last $\lambda(x) + 1$ requests of $(x, v)$. Thus, ALG uses at least 3 wavelengths for the request sequence. On the other hand, we have the following off-line algorithm. There is a set $\mathcal{A}$ of $\lambda(x)$ arc-disjoint $x$-arborescences in $G$ by Theorem II. For each of $\lambda(x)$ requests of $(x, v)$, we assign a dipath in distinct $x$-arborescence in $\mathcal{A}$, and assign the same wavelength, say $w$, to the dipaths. Since $\delta(x) > \lambda(x)$, there exists some $u_i$ $(1 \leqq i \leqq \delta(x))$ such that no dipaths above pass through $u_i$. Since $x$ is not a cut-vertex, there is a dipath $P(u_i, v)$ that dose not pass through $x$. For the remaining request of $(x, v)$, we assign a dipath consisting of arc $(x, u_i)$ and $P(u_i, v)$, and assign a wavelength different from $w$, say $w'$, to the dipaths. Then we can assign a dipath consisting of an arc $(x, u_j)$ with wavelength $w'$ for every requests $(x, u_j)$ $(j \neq i)$, and arc $(x, u_i)$ with wavelength $w$ for request $(x, u_i)$. In total, we use only 2 wavelengths for the request sequence, a contradiction. Thus we have the theorem. □

By Corollary 2 and Theorem 3 above, we have the following corollary.

**Corollary 4:** There is an on-line algorithm optimal for $G$ if and only if $\delta(x) = \lambda(x)$.

We can show a general lower bound as follows. Let $M$ be a mesh with $V(M) = \{0, 1, 2\}^2$. The vertices $ij$

and $i'j'$ are adjacent if and only if $|i-i'|+|j-j'|=1$. Let $G_M$ be the digraph associated with $M$.

**Theorem 4:** The competitive ratio of any on-line algorithm for $G_M$ is at least 4/3.

**Proof:** Let $u_1 = 01$, $u_2 = 10$, $u_3 = 12$, $u_4 = 21$, $v = 00$, and $x = 11$. Let ALG be any on-line algorithm for $G_M$. For any positive integer $l$, we consider the following sequence of $4l$ requests $I_{4l}$:

$$\underbrace{(x,u_1),\cdots,(x,u_1)}_{l},\underbrace{(x,u_2),\cdots,(x,u_2)}_{l},\underbrace{(x,u_3),\cdots,(x,u_3)}_{l},$$
$$\underbrace{(x,u_4),\cdots,(x,u_4)}_{l}. \quad (4)$$

If $\omega(G_M, I_{4l}, \mathrm{ALG}) \geqq 4l/3$ then we are done, because $\omega(G_M, I_{4l}) = l$ as easily seen, and we have

$$\omega(G_M, I_{4l}, \mathrm{ALG}) \geqq \frac{4}{3}l = \frac{4}{3}\omega(G_M, I_{4l}).$$

If $\omega(G_M, I_{4l}, \mathrm{ALG}) < 4l/3$ then we consider the following sequence of additional $4l$ requests $I'_{4l}$:

$$\underbrace{(x,v),(x,v),\cdots,(x,v)}_{4l}. \quad (5)$$

Suppose that ALG uses $l+i$ $(0 \leqq i < l/3)$ wavelengths for the sequence (4), and let $W = \{w_1, w_2, \cdots, w_{l+i}\}$ be the set of wavelengths used for the sequence (4). Since the outdegree of $x$ is 4, the maximum number of requests for which we can assign wavelengths in $W$ is $4(l+i)$. Since the number of requests in the sequence (4) is $4l$, ALG can use the wavelengths in $W$ for at most $4(l+i)-4l = 4i$ requests in the sequence (5). Since the indegree of $v$ is 2, ALG needs at least $(4l-4i)/2 = 2l-2i$ additional wavelengths not in $W$ for the sequence (5). Thus, ALG uses at least $(l+i)+(2l-2i) = 3l-i$ wavelengths for the concatenation of the sequences (4) and (5). Since $i < l/3$, we have

$$\omega(G_M, I_{4l} \cup I'_{4l}, \mathrm{ALG}) \geqq 3l - i > 3l - \frac{1}{3}l = \frac{8}{3}l.$$

On the other hand, it is easy to see that $\omega(G_M, I_{4l} \cup I'_{4l}) = 2l$. Thus we have

$$\omega(G_M, I_{4l} \cup I'_{4l}, \mathrm{ALG}) > \frac{4}{3}\omega(G_M, I_{4l} \cup I'_{4l}),$$

as desired. □

Notice that $\omega(G_M, I, \mathrm{ARB}) \leqq 2\omega(G_M, I)$ for any multicast $I$.

Our general upper bound for the competitive ratio is $\lceil \delta(x)/\lambda(x) \rceil$, and general lower bound is 4/3. It is an interesting open problem to close the gap between upper and lower bounds above.

## 4. Dynamic Multicasting

In the dynamic multicasting, a sequence of request arrivals and terminations is given for a multicast $I = \{(x,y)|y \in Y\}$. A dynamic algorithm assigns a dipath $P(x,y_i)$ and a wavelength to $P(x,y_i)$, so that no two dipaths sharing an arc are assigned the same wavelength if a request $(x,y_i)$ arrives, and deletes $P(x,y_i)$ together with the wavelength assigned if a request $(x,y_i)$ terminates. Let $I_j$ denote a collection of the existing requests just after $j$th request arrival or termination in the sequence. We denote by $\omega(G,x,L,\mathrm{ALG},I_j)$ the number of wavelengths used by a dynamic algorithm ALG for $I_j$ provided that $\mu(G,I_j) \leqq L$ for any $j$. Let $\omega(G,x,L,\mathrm{ALG})$ denote $\max_j \omega(G,x,L,\mathrm{ALG},I_j)$ and $\omega(G,x,L)$ denote the smallest $\omega(G,x,L,\mathrm{ALG})$ over all dynamic algorithms ALG. Notice that $\omega(G,x,L) \geqq L$.

Our dynamic algorithm ARB′ is obtained from ARB by simply adding the operation of deleting a path for the termination of the corresponding request. The following results are immediate from the corresponding results in the previous section.

**Theorem 5:**

$$\omega(G,x,L,\mathrm{ARB}') \leqq \left\lceil \frac{L \cdot \delta(x)}{\lambda(x)} \right\rceil.$$

**Corollary 5:** If $\delta(x) = O(1)$ then $\omega(G,x,L,\mathrm{ARB}') = O(L)$.

**Theorem 6:** $\omega(G,x,L) = L$ if and only if $\delta(x) = \lambda(x)$.

**Theorem 7:**

$$\omega(G_M,x,L) \geqq \frac{4}{3}L.$$

It should be noted that the performance of dynamic optical routing is considerably less than that of on-line optical routing in general, as mentioned in [6]. Our results indicate that the performance of dynamic multicasting is comparable to that of on-line multicasting.

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