Optimal Fault-Tolerant Linear Arrays

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ABSTRACT

This paper proves that for every positive integers n and k, we can explicitly construct a graph G with $n+O(k)$ vertices and maximum degree 3, such that even after removing any k vertices from G, the remaining graph still contains a path of length $n-1$. This settles a problem raised by Zhang [11, 12] in connection with the design of fault-tolerant linear arrays.

Categories and Subject Descriptors

C.1.4 [**Computer Systems Organization**]: Processor Architectures—*parallel architectures*; C.2.1 [**Computer Systems Organization**]: Computer Communication Networks—*network architecture and design*; F.1.2 [**Theory of Computation**]: Modes of Computation—*parallelism and concurrency*; G.1.0 [**Mathematics of Computing**]: Numerical Analysis—*General*; G.2.2 [**Mathematics of Computing**]: Discrete Mathematics—*Graph Theory*

General Terms

Algorithms, Design, Measurement, Performance, Reliability, Theory, Verification

Keywords

Fault-Tolerant Graph, Linear Array, Magnifier, Expander

1. INTRODUCTION

We consider the following problem motivated by the design of fault-tolerant linear array multiprocessor systems. Let G be a graph, and let $V(G)$ and $E(G)$ denote the vertex set and edge set of G, respectively. $\Delta(G)$ is the maximum degree of a vertex in G. For any $S \subseteq V(G)$, $G - S$ is the graph obtained from G by deleting the vertices of S together with the edges incident with the vertices in S. Let k be a positive integer. A graph G is called a k-FT (k-fault-tolerant) graph for a graph H if $G - F$ contains H as a subgraph for every $F \subseteq V(G)$ with $|F| \leq k$. Our problem is to construct a k-FT graph G for an n-vertex path P_n such that both $|V(G)|$ and $\Delta(G)$ are as small as possible.

A large amount of research has been devoted to constructing k-FT graphs for P_n [1–3, 6–8, 10–12]. Among others, Bruck, Cypher, and Ho [2] show a k-FT graph for P_n with $n + k^2$ vertices and maximum degree of 4. Zhang [11, 12] shows a k-FT graph for P_n with $n + O(k \log k)$ vertices and $O(\log k)$ maximum degree, and a k-FT graph for P_n with $n + O(k \log^2 k)$ vertices and $O(1)$ maximum degree. Zhang [11, 12] also raised the following open question: Is it possible to construct an explicit k-FT graph for P_n with $n + O(k)$ vertices and $O(1)$ maximum degree? It should be noted that such a k-FT graph is optimal in the sense that every k-FT graph for P_n has $n + \Omega(k)$ vertices and $\Omega(1)$ maximum degree.

In this paper, we settle the question by showing the following.

THEOREM 1. *For any positive integers n and k, we can explicitly construct a k-FT graph G for P_n such that $|V(G)| = n + O(k)$ and $\Delta(G) = 3$.* □

We note that Alon and Chung [1] proved that for any positive integers n and $k = \Omega(n)$, we can explicitly construct a k-FT graph G for P_n such that $|V(G)| = n + O(k)$ and $\Delta(G) = O(1)$.

2. PROOF OF THEOREM 1

Let $\Gamma_G(v)$ denote the set of vertices adjacent to v in a graph G, $\Gamma_G(X) = \bigcup_{v \in X} \Gamma_G(v)$, and $\partial X = \Gamma_G(X) - X$ for any $X \subseteq V(G)$. We define that $\deg_G(v) = |\Gamma_G(v)|$, and $\Delta(G) = \max_{v \in V(G)} \deg_G(v)$.

In order to prove Theorem 1, we first need a few results on expanders and magnifiers.

2.1 Expanders

Let $c \leq 1$. A bipartite graph B with bipartition (I, O) is an (n, d, c)-expander if the following three conditions are satisfied:

1. $|I| = |O| = n$;
2. $\Delta(B) \leq d$;
3. $|\Gamma_B(X)| \geq \left\{1 + c\left(1 - \frac{|X|}{n}\right)\right\} \cdot |X|$ for every $X \subseteq I$.

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For any positive integer m, let $[m] = \{0, 1, \ldots, m-1\}$. $\mathrm{GG}(m)$ is the bipartite graph with bipartition (I, O) defined as follows: $I = [m]^2 \times \{0\}$ and $O = [m]^2 \times \{1\}$; Each vertex $[i, j, 0] \in I$ is connected with seven vertices $[i, j, 1], [i + 2j, j, 1], [i + 2j + 1, j, 1], [i + 2j + 2, j, 1], [i, j + 2i, 1], [i, j + 2i + 1, 1], [i, j + 2i + 2, 1] \in O$, each by an edge, where additions are performed modulo m. Gabber and Galil proved in [5] the following theorem.

Theorem I. [5] *For any positive integer m, $\mathrm{GG}(m)$ is an $(m^2, 7, (2-\sqrt{3})/2)$-expander.* □

2.2 Magnifiers

Let $c \le 1$. A graph G is an (n, d, c)-magnifier if the following three conditions are satisfied:

1. $|V(G)| = n$;
2. $\Delta(G) \le d$;
3. $|\partial X| \ge c|X|$ for every $X \subset V(G)$ with $|X| \le n/2$.

Note that an (n, d, c)-magnifier is connected if $c > 0$.

For any positive integer m, $M(m)$ is the graph obtained from $\mathrm{GG}(m)$ by merging $[i, j, 0]$ and $[i, j, 1]$ for every $i, j \in [m]$ and removing all self-loops, that is the graph defined as follows: $V(M(m)) = [m]^2$; Each vertex $[i, j] \in V(M(m))$ is connected with 12 vertices $[i \pm 2j, j], [i \pm (2j+1), j], [i \pm (2j+2), j], [i, j \pm 2i], [i, j \pm (2i+1)], [i, j \pm (2j+2)]$, each by an edge, where additions are performed modulo m.

Lemma 1. *For any positive integer m, $M(m)$ is an $(m^2, 12, (2-\sqrt{3})/4)$-magnifier.*

Proof. Fix any $X \subseteq V(M(m))$ with $|X| \le m^2/2$, and let $X' = \{[i, j, 0] : [i, j] \in X\}$. Since $\mathrm{GG}(m)$ is an $(m^2, 7, (2-\sqrt{3})/2)$-expander by Theorem I and $|X'| = |X| \le m^2/2$, we have

$$\begin{aligned}
|\partial X| &\ge |\Gamma_{GG(m)}(X')| - |X'| \\
&\ge \left\{1 + \frac{2-\sqrt{3}}{2}\left(1 - \frac{|X'|}{m^2}\right)\right\} \cdot |X'| - |X'| \\
&\ge \frac{2-\sqrt{3}}{4} \cdot |X'| \\
&\ge \frac{2-\sqrt{3}}{4} \cdot |X|.
\end{aligned}$$

Hence, $M(m)$ is an $(m^2, 12, (2-\sqrt{3})/4)$-magnifier. □

Lemma 2. *If G is an (n, d, c)-magnifier and $k \le cn/4$ is a positive integer then $G - F$ contains a connected component of size at least $n - (1 + 1/c)k$ for any $F \subset V(G)$ with $|F| \le k$.*

Proof. Fix any set $F \subset V(G)$ with $|F| \le k \le cn/4$. Let $G_0, G_1, \ldots, G_{t-1}$ be the connected components of $G - F$, and $X_i = V(G_i)$ for any $i \in [t]$.

Claim 1. *For any $S \subseteq [t]$,*

$$\sum_{i \in S} |X_i| \le \frac{k}{c} \quad \text{or} \quad \sum_{i \in S} |X_i| > \frac{n}{2}.$$

Proof of Claim 1: Assume contrary that there exists a set $S \subseteq [t]$ such that

$$\frac{k}{c} < \sum_{i \in S} |X_i| \le \frac{n}{2},$$

and let $X = \bigcup_{i \in S} X_i$. Then, we have

$$\frac{k}{c} < |X| = \left|\bigcup_{i \in S} X_i\right| = \sum_{i \in S} |X_i| \le \frac{n}{2},$$

and so $|\partial X| \ge c|X| > k$. On the other hand, since $\partial X \subseteq F$, we have $|\partial X| \le |F| \le k$, which is a contradiction. □

The following is immediate from Claim 1.

Claim 2. *$|X_i| \le k/c$ or $|X_i| > n/2$ for any $i \in [t]$.* □

Claim 3. *If $S = \{i \in [t] : |X_i| \le k/c\}$ then*

$$\left|\bigcup_{i \in S} X_i\right| \le \frac{k}{c}.$$

Proof of Claim 3: Assume for contradiction that

$$\left|\bigcup_{i \in S} X_i\right| > \frac{k}{c},$$

and let $S = \{i_1, i_2, \ldots, i_s\}$, where $s = |S|$. By the assumption, there exists a smallest integer $l \le s$ such that

$$\left|\bigcup_{j=1}^{l} X_{i_j}\right| > \frac{k}{c}.$$

By the definition of l, we have

$$\left|\bigcup_{j=1}^{l-1} X_{i_j}\right| \le \frac{k}{c}.$$

By the definition of S, we have

$$|X_{i_l}| \le \frac{k}{c}.$$

Since $k \le cn/4$ by the assumption of Lemma 2, we conclude that

$$\frac{k}{c} < \left|\bigcup_{j=1}^{l} X_{i_j}\right| \le \frac{2k}{c} \le \frac{n}{2},$$

which is contradicting to Claim 1. □

By Claims 2 and 3, there exists a unique integer i such that $|X_i| > n/2$ since $k + (k/c) \le 2k/c \le n/2$. Thus, we have

$$|X_i| = |V(G)| - |F| - \left|\bigcup_{j \ne i} X_j\right| \ge n - \left(1 + \frac{1}{c}\right) k,$$

and we conclude that G_i is a connected component of size at least $n - (1 + 1/c)k$. This completes the proof of Lemma 2. □

2.3 Products of Magnifiers and Paths

For any two graphs G and H, the product of G and H, denoted by $G \times H$, is the graph defined as follows: $V(G \times H) = V(G) \times V(H)$; Any two vertices $[u, x]$ and $[v, y]$ in $G \times H$ are joined by an edge if one of the following conditions is satisfied:

1. $(u, v) \in E(G)$ and $x = y$, or
2. $u = v$ and $(x, y) \in E(H)$.

LEMMA 3. *Let n_1 and n_2 be two positive integers, and k be a positive integer with $k \le \min\{n_1/4, n_2 - 1\}$. If G is an (n_1, d, c)-magnifier for some positive integer d and positive number c then $G \times P_{n_2} - F$ contains a connected component of size at least $n - (1 + 1/c)k$ for any $F \subseteq V(G \times P_{n_2})$ with $|F| = k$, where $n = n_1 n_2$ is the number of vertices in $G \times P_{n_2}$.*

PROOF. Let $S = \{v \in V(G) : (\{v\} \times V(P_{n_2})) \cap F = \emptyset\}$, $F_x = \{v \in V(G) : [v, x] \in F\}$, and $k_x = |F_x|$ for any $x \in V(P_{n_2})$. Then, $|S| \ge n_1 - k \ge (3 - c)n_1/4 > n_1/2$ and $k = \sum_{v \in V(P_{n_2})} k_x$. By Lemma 2, $G - F_x$ contains a connected component of size at least $n_1 - (1 + 1/c)k_x$ if $k_x \le cn_1/4$. We denote the connected component by G_x. For any $x \in V(P_{n_2})$, V_x is defined as the set of vertices in G_x if $k_x \le cn_1/4$, and $\emptyset$ otherwise. Let

$$U = \bigcup_{x \in V(P_{n_2})} (S \cup V_x) \times \{x\}$$

and Q denote the subgraph of $G \times P_{n_2}$ induced by U. We are going to show that the connected component containing Q is a desired one by proving that Q is connected and $|U| \ge n + (1 + 1/c)k$.

We need a few claims in order to prove that Q is connected.

CLAIM 4. *$V_z = V(G)$ for some $z \in V(P_{n_2})$.*

PROOF OF CLAIM 4: For otherwise, $k_x \ge 1$ for all $x \in V(P_{n_2})$, and hence $k = \sum_{x \in V(P_{n_2})} k_x \ge n_2 \ge k + 1$, which is a contradiction. □

CLAIM 5. *$S \cap V_x \ne \emptyset$ for any $x \in V(P_n)$ with $k_x \le cn_1/4$.*

PROOF OF CLAIM 5: The claim follows from the facts that $|S| > n_1/2$ and

$$|V_x| \ge n_1 - \left(1 + \frac{1}{c}\right) k_x \ge \frac{3 - c}{4} \cdot n_1 > \frac{n_1}{2}. \qquad \square$$

Now we are ready to prove that Q is connected. We show that for any two vertices $[u, x], [v, y] \in U$, there exists a path in Q connecting them.

Consider the case when $x \ne y$, $k_x \le cn_1/4$, $k_y \le cn_1/4$, $u \in V_x - S$, and $v \in V_y - S$. Since $u' \in S \cap V_x$ and $v' \in S \cap V_y$ by Claim 5, we conclude that $[u, x]$ and $[u, y]$ are connected by the concatenation of the following five paths: (i) a path connecting $[u, x]$ and $[u', x]$ on $G_x \times \{x\}$; (ii) the path connecting $[u', x]$ and $[u', z]$ on $\{u'\} \times P_{n_2}$; (iii) a path connecting $[u', z]$ and $[v', z]$ on $G \times \{z\}$; (iv) the path connecting $[v', z]$ and $[v', y]$ on $\{v'\} \times P_{n_2}$; and (v) a path connecting $[v', y]$ and $[v, y]$ on $G_y \times \{y\}$, where z is a vertex satisfying the condition in Claim 4. (See Figure 1.)

For the remaining cases, it is easy to show that there exists a path connecting the vertices $[u, x]$ and $[v, y]$ by similar arguments. Thus, we conclude that Q is connected.

It remains to show that $|U| \ge n - (1 + 1/c)k$. If $k_x \le cn_1/4$ then we have

$$|V_x| \ge n_1 - (1 + 1/c)k_x,$$

and if $k_x > cn_1/4$ then we have

$$\begin{aligned} |S| &\ge n_1 - k \ge n_1 - (n_1/4) \\ &> n_1 - (k_x/c) \ge n_1 - (1 + 1/c)k_x. \end{aligned}$$

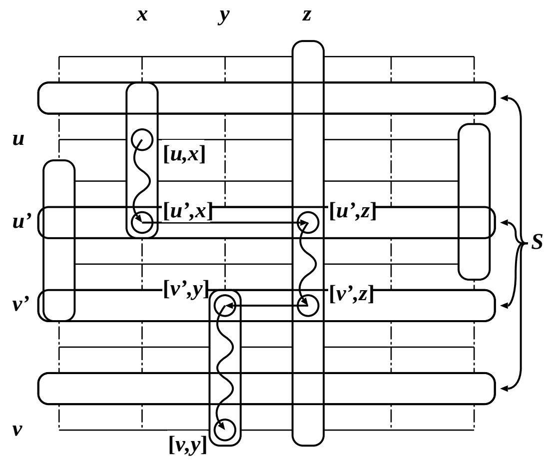

Figure 1: Path connecting $[u, x]$ and $[v, y]$

Since

$$U = \bigcup_{x \in V(P_{n_2})} (S \cup V_x) \times \{x\},$$

we have

$$\begin{aligned} |U| &\ge \sum_{x \in V(P_{n_2})} \left\{ n_1 - \left(1 + \frac{1}{c}\right) k_x \right\} \\ &= n - \left(1 + \frac{1}{c}\right) k. \qquad \square \end{aligned}$$

2.4 Proof of Theorem 1

We first prove the following lemma.

LEMMA 4. *For any positive integers n and k, we can construct a graph $H_{n,k}$ satisfying the following three conditions:*

(c1) $H_{n,k} - F$ contains a connected component of size at least n for any $F \subseteq V(H_{n,k})$ with $|F| \le k$,

(c2) $|V(H_{n,k})| \le n + \gamma k + \delta$ for some constants γ and δ, and

(c3) $\Delta(H_{n,k}) \le 14$.

PROOF. First, assume that $1 \le k \le \sqrt{n/8}$. Set m as an integer satisfying that $(m - 1)^2 < 4k \le m^2$, and $c = (2 - \sqrt{3})/4$. Then, we have

$$4k \le m^2 < 4k + 4\sqrt{k} + 1 \le 8k + 1.$$

Let $n_1 = m^2$ and $n_2 = \lceil (n + (1 + 1/c)k)/m^2 \rceil$. Then, we have

$$k \le \frac{n_1}{4}, \tag{1}$$

and

$$n_1 n_2 \ge n + \left(1 + \frac{1}{c}\right) k. \tag{2}$$

Since

$$\begin{aligned} n_2 &= \left\lceil \frac{n + (1 + 1/c)k}{m^2} \right\rceil \ge \left\lceil \frac{8k^2 + (1 + 1/c)k}{8k + 1} \right\rceil \\ &= \left\lceil k + \frac{k}{c(8k + 1)} \right\rceil \ge k + 1, \end{aligned}$$

we have

$$k \le n_2 - 1. \tag{3}$$

We show that $M(m) \times P_{n_2}$ is a desired graph $H_{n,k}$. Recall that $M(m)$ is a $(n_1, 12, c)$-magnifier by Lemma 1. It follows that $\Delta(M(m) \times P_{n_2}) = 14$. Thus, $M(m) \times P_{n_2}$ satisfies (c3). Since

$$\begin{aligned} |V(M(m) \times P_{n_2}) &= n_1 n_2 \\ &\le \left\{ \frac{n + (1 + 1/c)k}{n_1} + 1 \right\} n_1 \\ &= n + \left(1 + \frac{1}{c}\right) k + n_1 \\ &\le n + \left(9 + \frac{1}{c}\right) k + 1, \end{aligned}$$

$M(m) \times P_{n_2}$ satisfies (c2). From inequalities (1), (2), and (3) together with Lemma 3, $M(m) \times P_{n_2}$ satisfies (c1). Hence, $M(m) \times P_{n_2}$ is a desired graph $H_{n,k}$.

Next, assume that $\sqrt{n/8} < k \le cn/(3-c)$, where $c = (2 - \sqrt{3})/4$. Set m as an integer satisfying that

$$(m-1)^2 < n + \left(1 + \frac{1}{c}\right) k \le m^2.$$

We show that $M(m)$ is a desired graph $H_{n,k}$. Since $M(m)$ is a $(n_1, 12, c)$-magnifier and $\Delta(M(m)) = 12$, $M(m)$ satisfies (c3). Since

$$\begin{aligned} &|V(M(m))| = m^2 \\ &< \quad n + \left(1 + \frac{1}{c}\right) k + 2\sqrt{n + \left(1 + \frac{1}{c}\right) k} + 1 \\ &< \quad n + \left(1 + \frac{1}{c}\right) k + 2\sqrt{8k^2 + \left(1 + \frac{1}{c}\right) k} + 1 \\ &\le \quad n + \left(1 + \frac{1}{c} + 2\sqrt{9 + \frac{1}{c}}\right) k + 1, \end{aligned}$$

$M(m)$ satisfies (c2). Since

$$\frac{c}{4} \left\{ n + \left(1 + \frac{1}{c}\right) k \right\} - k = \frac{cn}{4} - \frac{3-c}{4} k \ge 0,$$

we have

$$k \le \frac{c}{4} \left\{ n + \left(1 + \frac{1}{c}\right) k \right\} \le \frac{cm^2}{4}.$$

Thus, by Lemma 2, $M(m)$ satisfies (c1). Hence, we conclude that $M(m)$ is a desired graph $H_{n,k}$.

Finally, assume that $k > cn/(3-c)$, where $c = (2-\sqrt{3})/4$. Set $n' = \lceil (3-c)k/c \rceil$. Since $k \le cn'/(3-c)$, we can construct $H_{n',k}$ as shown above. We show that $H_{n',k}$ is a desired graph. Since $H_{n',k} - F$ contains a connected component of size at least $n' \ge n$ for any $F \subseteq V(H_{n',k})$ with $|F| \le k$, $H_{n',k}$ satisfies (c1). Since $|V(H_{n',k})| \le n' + \gamma' k + \delta' \le n + \{\gamma' + (3-c)/c\} k + (\delta' + 1)$ for some γ' and δ', $H_{n',k}$ satisfies (c2). Since $\Delta(H_{n',k}) \le 14$, $H_{n',k}$ satisfies (c3). Thus, $H_{n',k}$ is a desired graph. $\square$

Now, we are ready to prove Theorem 1. Let $d = 14$, $n' = \lceil n/2d \rceil$, and f_u be a one-to-one mapping from $\Gamma_{H_{n',k}}(u)$ to $[d]$. $G_{n,k}$ is the graph defined as follows: $V(G_{n,k}) = V(H_{n',k}) \times [2d]$; Any two vertices $[u,i], [v,j] \in V(G_{n,k})$ are connected by an edge if one of the following two conditions is satisfied:

(i) $u = v$ and $j = (i \pm 1) \bmod (2d)$;

(ii) $(u,v) \in E(H_{n',k})$, $i = 2f_u(v) + r$, $j = 2f_v(u) + r$, and $r \in [2]$.

We are going to show that $G_{n,k}$ is a desired k-FT graph for P_n. It is easy to see the following two lemmas.

LEMMA 5. *$|V(G_{n,k})| \le n + 2d\gamma k + 2d$.* $\square$

LEMMA 6. *$\Delta(G_{n,k}) = 3$.* $\square$

It remains to show the following:

LEMMA 7. *$G_{n,k}$ is a k-FT graph for P_n.*

PROOF. We show that for any $F \subseteq V(G_{n,k})$ with $|F| \le k$, $G_{n,k} - F$ contains P_n as a subgraph. Let $F' = \{v \in V(H_{n',k}) : [v,j] \in F, j \in [2d]\}$. Since $|F'| \le |F| \le k$ by definition, $H_{n',k} - F'$ contains a connected component $\mathcal{H}$ of size at least n'. Let T denote a spanning tree of $\mathcal{H}$. A vertex r of T is designated as a root, and T is considered as a rooted tree. For any $v \in V(T)$, let $T(v)$ is a subtree of T consisting of the descendants of v. Define that

$$\begin{aligned} X(v) &= \{[v,j] : j \in [2d]\}, \\ Y(v) &= \{[u,i] : u \in T(v), i \in [2d]\}, \end{aligned}$$

and $\mathcal{G}(v)$ denote the subgraph of $G_{n,k}$ induced by $Y(v)$.

CLAIM 6. *Let $v_0, \ldots, v_{m-1}$ be the children of $u \in V(T)$. If $\mathcal{G}(v_l)$ has a Hamilton cycle for every $l \in [m]$ then $\mathcal{G}(u)$ has a Hamilton cycle.*

PROOF OF CLAIM 6: For each $l \in [m]$, let C^l denote a Hamilton cycle of $\mathcal{G}(v_l)$, and let $C(u)$ denote the subgraph of $G_{n,k}$ induced by $X(u)$, which is isomorphic to C_{2d}. Define C as the graph obtained from $C^0, C^1, \ldots, C^{m-1}$, and $C(u)$ by replacing two edges $([u, 2f_u(v_l)], [u, 2f_u(v_l)+1])$ and $([v_l, 2f_{v_l}(u)], [v_l, 2f_{v_l}(u)+1])$ with $([u, 2f_u(v_l)], [v_l, 2f_{v_l}(u)])$ and $([u, 2f_u(v_l)+1], [v_l, 2f_{v_l}(u)+1])$ for each $l \in [m]$. It is easy to see that C is a Hamilton cycle of $\mathcal{G}(u)$. (See Figure 2.) $\square$

It is easy to see that $\mathcal{G}(v)$ has a Hamilton cycle if $v \in V(T)$ is a leaf. Hence, we have by Claim 6 a Hamilton cycle of $\mathcal{G}(r)$. Since

$$|V(\mathcal{G}(r))| = 2d \cdot |V(T)| \ge 2dn' \ge 2d \cdot \frac{n}{2d} = n,$$

$G_{n,k} - F$ contains P_n as a subgraph. Hence, we conclude that $G_{n,k}$ is a k-FT graph for P_n.

Lemmas 5, 6, and 7 complete the proof of Theorem 1. $\square$

3. CONCLUDING REMARKS

It is worth noting that there exists no k-FT graph for P_n with $n + O(k)$ vertices and maximum degree of 2. Let G be a k-FT graph for P_n with maximum degree of 2, and let N denote the number of vertices in G. Assume without loss of generality that G has no connected components of

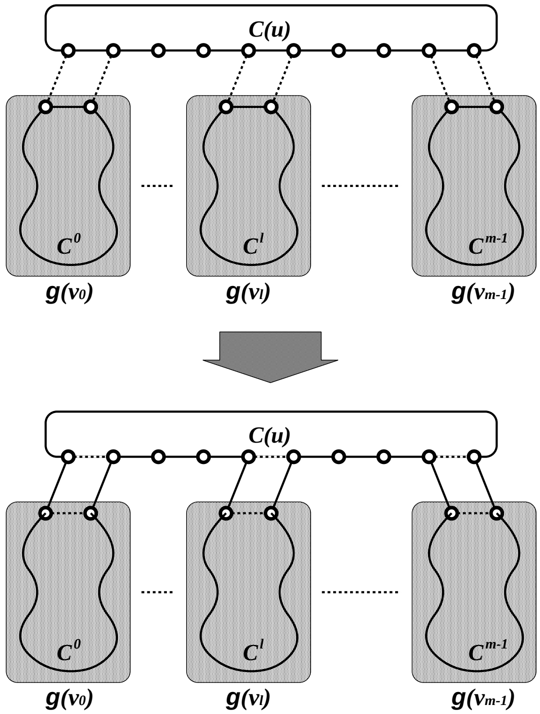

Figure 2: Construction of C from $m+1$ cycles $C^0, \ldots, C^{m-1}$, and $C(u)$.

size smaller than n. Let α be the number of connected components in G. Then,

$$N \geq \alpha n. \tag{4}$$

Since the maximum degree of G is 2, a graph H obtained from G by removing a vertex of each connected component in G is a disjoint union of paths, which is a subgraph of $P_{N-\alpha}$. If G is a k-FT graph for P_n, H is a $(k-\alpha)$-FT graph for P_n, and so $P_{N-\alpha}$ is also $(k-\alpha)$-FT graph for P_n. Hence,

$$N - \alpha \geq (k - \alpha + 1)n,$$

that is

$$N \geq (k - \alpha + 1)n + \alpha. \tag{5}$$

By inequalities (4) and (5), we have

$$N \geq \frac{\alpha n + (k - \alpha + 1)n + \alpha}{2} \geq \frac{(k+1)n}{2}.$$

It follows that there exists no k-FT graph for P_n with $n + O(k)$ vertices and maximum degree of 2.

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