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Sparse networks tolerating random faults

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Abstract

A network G^* is called random-fault-tolerant (RFT) network for a network G if G^* contains a fault-free isomorphic copy of G with high probability even if each processor fails independently with constant probability. This paper proposes a general method to construct an RFT network G^* for any network G with N processors such that G^* has $O(N)$ processors. Based on the construction, we also show that if G is a Cayley, de Bruijn, shuffle-exchange, or partial k-tree network with N processors and M communication links then we can construct an RFT network for G with $O(N)$ processors and $O(M \log N)$ communication links. Cayley networks contain many popular networks such as circulant, hypercube, CCC, wrapped butterfly, star, and pancake networks.

Keywords: Fault-tolerant network; Cayley graph; de Bruijn network; Shuffle-exchange network; Partial k-tree

1. Introduction

This paper considers the following problem in connection with the design of fault-tolerant interconnection networks for multiprocessor systems: given an N-vertex graph G, construct an $O(N)$-vertex graph $\tilde{G}$ with a minimum number of edges such that even after deleting each vertex from $\tilde{G}$ independently with constant probability, the remaining graph contains G as a subgraph, with probability converging to 1, as $N \to \infty$. An $O(N)$-vertex graph G^* is called an random-fault-tolerant (RFT) graph for an N-vertex graph G if G^* contains G as a subgraph with probability converging to 1, as $N \to \infty$, even after deleting each vertex from G^* independently with constant probability. Therefore, our problem is to find an RFT graph with a minimum number of edges for a given graph.

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Let $V(G)$ and $E(G)$ be the vertex set and edge set of a graph G, respectively. Fraigniaud et al. showed in [5] that for any N-vertex graph G, there exists an RFT graph for G with $O(|E(G)|\log^2 N)$ edges, and that for any N-vertex graph with $O(N)$ edges and maximum degree of $\Omega(N)$, any RFT graph for G has $\omega(|E(G)|)$ edges. It is an interesting open problem posed in [5] to decide whether any N-vertex graph G has an RFT graph with $O(|E(G)|\log N)$ edges.

It is also shown in [5] that if G is an N-vertex tree then we can construct an RFT graph with $O(|E(G)|\log N)$ edges. Friedman and Pippenger showed in [6] that if G is a tree with bounded vertex degree then we can construct an RFT graph with $O(|E(G)|)$ edges. It is further shown in [5] that if G is a path or cycle then we can construct an RFT graph with $O(|E(G)|)$ edges. The result for paths was also shown by Alon and Chung in [3]. Tamaki showed in [8] that if G is an N-vertex mesh or torus then we can construct an RFT graph with $O(|E(G)|\log\log N)$ edges.

In this paper, we propose a general method to construct an RFT graph for any graph. Based on the construction, we show that if G is an N-vertex Cayley graph, de Bruijn graph, shuffle-exchange graph, or partial k-tree, we can construct an RFT graph for G with $O(|E(G)|\log N)$ edges. Cayley graphs contain many popular networks such as circulant, hypercube, CCC, wrapped butterfly, star, and pancake graphs. Our result for partial k-trees is a natural generalization of a result for trees in [5].

2. General construction

For any positive integer l, let $[l]=\{0,1,\ldots,l-1\}$. For any set of S, a collection $\mathscr{S}=\{S_0,S_1,\ldots,S_{l-1}\}$ of subsets of S is a partition of S if $\bigcup_{i\in[l]} S_i=S$ and $S_i\cap S_j=\emptyset$ for any $i\neq j$.

Let G be any N-vertex graph. For any partition $\mathscr{V}=\{V_0,V_1,\ldots,V_{l-1}\}$ of $V(G)$, define

$$\Lambda(G,\mathscr{V})=\{(i,j)\colon \exists(u,v)\in E(G)(u\in V_i,\ v\in V_j)\},\quad \text{and}$$

$$\lambda(G,\mathscr{V})=|\Lambda(G,\mathscr{V})|.$$

Let $0<p<1$ be the probability for each vertex to be deleted. The deleted and undeleted vertices are said to be faulty and fault-free, respectively.

Let $\mathscr{V}=\{V_0,V_1,\ldots,V_{l-1}\}$ be any partition of $V(G)$ such that $|V_i|\leqslant\alpha\ln N$ for any $i\in[l]$ and $l\leqslant\beta N/\ln N$ for some fixed positive numbers α and β. Let $V_0^*,V_1^*,\ldots,V_{l-1}^*$ be l sets such that $|V_i^*|=\lceil\gamma\ln N\rceil$ for any $i\in[l]$ and $V_i^*\cap V_j^*=\emptyset$ for any $i\neq j$, where

$$\gamma=\frac{(\sqrt{2\alpha+1}+1)^2}{2(1-p)}.$$

Note that γ is fixed since α and p are fixed. Then, $G^*[\mathscr{V}]$ is the graph defined as follows:

$$V(G^*[\mathscr{V}])=V_0^*\cup V_1^*\cup\cdots\cup V_{l-1}^*;$$

$$E(G^*[\mathscr{V}])=\{(u^*,v^*)\colon u^*\in V_i^*,\ v^*\in V_j^*,\ (i,j)\in\Lambda(G,\mathscr{V})\}.$$

Theorem 1. *Let G be any N-vertex graph, and let $\mathscr{V} = \{V_0, V_1, \ldots, V_{l-1}\}$ be any partition of $V(G)$ such that $|V_i| = \mathrm{O}(\ln N)$ and $l = \mathrm{O}(N/\ln N)$. Then $G^*[\mathscr{V}]$ is an RFT graph for G with $\mathrm{O}(\lambda(G, \mathscr{V}) \log^2 N)$ edges.*

Proof. We prove the theorem by a series of lemmas. It is easy to see the following two lemmas.

Lemma 1. $|V(G^*[\mathscr{V}])| \leqslant \beta N/\ln N \cdot \lceil \gamma \ln N \rceil$.

Lemma 2. $|E(G^*[\mathscr{V}])| \leqslant \lambda(G, \mathscr{V}) \cdot \lceil \gamma \ln N \rceil^2$.

Now we prove that $G^*[\mathscr{V}]$ is an RFT graph for G. We need a few probabilistic notations and lemmas.

For any event E, let Prob$[E]$ denote the probability of E. For any random variable X and real number r, let $\{X \leqslant r\}$ denote the event that $X \leqslant r$. The probability of $\{X \leqslant r\}$ is denoted by Prob$[X \leqslant r]$ instead of Prob$[\{X \leqslant r\}]$. The following inequality is well-known as Chernoff Bound.

Lemma 3 (Hagerup and Rüb [7]). *Let X be the binomial variable with parameters m and q, that is, the number of successes in m Bernoulli trials with probabilities q for success and $1 - q$ for failure. Then, for any constant $0 < \varepsilon < 1$,*

$$\mathrm{Prob}[X \leqslant (1-\varepsilon)qm] \leqslant \exp(-\tfrac{1}{2}\varepsilon^2 qm).$$

Lemma 4. *Let Y_i be the number of fault-free vertices of V_i^*. Then, for any $i \in [l]$,*

$$\mathrm{Prob}[Y_i \leqslant \alpha \ln N] \leqslant \frac{1}{N}.$$

Moreover,

$$\mathrm{Prob}\left[\bigcup_{i=0}^{l-1} \{Y_i \leqslant \alpha \ln N\}\right] \leqslant \frac{\beta}{\ln N}.$$

Proof. Set $\varepsilon = 2/(\sqrt{2\alpha+1}+1)$, $q = 1 - p$, and $m = \lceil \gamma \ln N \rceil$. Since $0 < \varepsilon < 1$,

$$\begin{aligned}(1-\varepsilon)qm &\geqslant \frac{\sqrt{2\alpha+1}-1}{\sqrt{2\alpha+1}+1}(1-p)\frac{(\sqrt{2\alpha+1}+1)^2}{2(1-p)}\ln N \\ &= \frac{2\alpha}{(\sqrt{2\alpha+1}+1)^2}\,\frac{(\sqrt{2\alpha+1}+1)^2}{2}\ln N \\ &= \alpha \ln N,\end{aligned}$$

and

$$\frac{1}{2}\varepsilon^2 qm \geqslant \frac{1}{2}\,\frac{4}{(\sqrt{2\alpha+1}+1)^2}(1-p)\frac{(\sqrt{2\alpha+1}+1)^2}{2(1-p)}\ln N = \ln N,$$

we obtain by Lemma 3 that

$$\mathrm{Prob}[Y_i \leqslant \alpha \ln N] \leqslant \mathrm{Prob}[Y_i \leqslant (1-\varepsilon)qm] \leqslant \exp\left(-\frac{1}{2}\varepsilon^2 qm\right) \leqslant \frac{1}{N}.$$

Moreover,

$$\mathrm{Prob}\left[\bigcup_{i=0}^{l-1}\{Y_i \leqslant \alpha \ln N\}\right] \leqslant \sum_{i=0}^{l-1} \mathrm{Prob}[Y_i \leqslant \alpha \ln N] \leqslant \frac{\beta}{\ln N}. \qquad \square$$

Lemma 5. *$G^*[\mathscr{V}]$ is an RFT graph for G.*

Proof. Let ϕ be a one-to-one mapping from $V(G)$ to $V(G^*[\mathscr{V}])$ such that $\phi(v)$ is a fault-free vertex of V_i^* for any $v \in V_i$. By Lemma 4, such ϕ exists with probability at least $1-(\beta/\ln N)$.

Now we show that $(\phi(u), \phi(v)) \in E(G^*[\mathscr{V}])$ for any $(u,v) \in E(G)$. Let $u \in V_i$ and $v \in V_j$. Then, $(i,j) \in \Lambda(G, \mathscr{V})$. Since $\phi(u) \in V_i^*$ and $\phi(v) \in V_j^*$, we conclude that $(\phi(u), \phi(v)) \in E(G^*[\mathscr{V}])$. Hence $G^*[\mathscr{V}]$ is an RFT graph for G. $\square$

This completes the proof of Theorem 1. $\square$

Since $\lambda(G, \mathscr{V}) \leqslant |E(G)|$, we obtain the following corollary.

Corollary 1. *Let G be any N-vertex graph, and let $\mathscr{V} = \{V_0, V_1, \ldots, V_{l-1}\}$ be any partition of $V(G)$ such that $|V_i| = \mathrm{O}(\ln N)$ and $l = \mathrm{O}(N/\ln N)$. $G^*[\mathscr{V}]$ is an RFT graph for G with $\mathrm{O}[(E(G)|\log^2 N)$ edges.*

This corollary means that, for any N-vertex graph G, there exists an RFT graph for G with $\mathrm{O}(|E(G)|\log^2 N)$ edges, which was first obtained in [5].

3. RFT graphs for Cayley graphs

3.1. Groups and Cayley graphs

A group is a set Γ together with a binary operation satisfying the following three conditions: (i) $(xy)z = x(yz)$ for all $x,y,z \in \Gamma$, (ii) there exists $e \in \Gamma$, which is called the unit element of Γ, such that $xe = ex = x$ for all $x \in \Gamma$, and (iii) for every $x \in \Gamma$, there exists $y \in \Gamma$, which is called the inverse for x, such that $xy = yx = e$. The inverse for x is denoted by x^{-1}. Γ is said to be finite if $|\Gamma|$ is finite, and Γ is said to be abelian if $xy = yx$ for all $x, y \in \Gamma$. A subset Γ' of Γ is called a subgroup of Γ if the following two conditions are satisfied: (i) $xy \in \Gamma'$ for all $x, y \in \Gamma'$, and (ii) $x^{-1} \in \Gamma'$ for all $x \in \Gamma'$.

Let Γ be a group and let $S \subset \Gamma$ such that $e \notin S$ and if $s \in S$ then $s^{-1} \in S$. The Cayley graph $C(\Gamma, S)$ is defined as follows: $V(C(\Gamma,S)) = \Gamma$; Any two vertices $x, y \in \Gamma$ are connected by an edge if $y = xs$ for some $s \in S$. $C(\Gamma, S)$ is said to be abelian if Γ is abelian. Cayley graphs are extensively considered in [2] as symmetric interconnection networks.

We denote $|\Gamma|$ by N throughout this section. It is easy to see the following.

Lemma 6. $|E(C(\Gamma,S))| = |S| \cdot N/2$.

3.2. RFT graphs for Abelian Cayley graphs

Let Γ_1 and Γ_2 be two groups. A mapping $f : \Gamma_1 \to \Gamma_2$ is called a homomorphism if $f(xy)=f(x)f(y)$ for any $x, y \in \Gamma_1$. If f is a bijection then f is called an isomorphism. Γ_1 is isomorphic to Γ_2, denoted by $\Gamma_1 \simeq \Gamma_2$, if there exists an isomorphism of Γ_1 into Γ_2.

The direct product of two groups Γ_1 and Γ_2 is the set $\Gamma_1 \times \Gamma_2$ together with a binary operation defined as follows: For any $[x_1,x_2],[y_1,y_2] \in \Gamma_1 \times \Gamma_2$,

$$[x_1,x_2][y_1,y_2] = [x_1 y_1, x_2 y_2].$$

The direct product of Γ_1 and Γ_2 is a group such that the unit element of the group is $[e_1,e_2]$, and $[x_1,x_2]^{-1} = [x_1^{-1},x_2^{-1}]$, where e_i is the unit element of $\Gamma_i (i = 1,2)$.

For any positive integer m, let $\oplus_m$ denote addition modulo m. Note that $[m]$ with $\oplus_m$ is an abelian group. The following theorem is well-known. See, for example, [1] for the proof.

Theorem I. *If Γ is a finite abelian group then*

$$\Gamma \simeq [m_1] \times [m_2] \times \cdots \times [m_n]$$

for some integers $m_1,m_2,\dots,m_n \geqslant 2$, with $N = m_1 m_2 \cdots m_n$.

Let Γ be an abelian group represented as $[m_1] \times [m_2] \times \cdots \times [m_n]$. Let d be the least integer such that $m_1 m_2 \cdots m_d \geqslant \log N$. Let δ be the least integer such that $\delta \cdot m_1 \cdots m_{d-1} \geqslant \log N$ if $d \geqslant 2$, and $\delta \geqslant \log N$ if $d = 1$. Let $m' = \lceil m_d/\delta \rceil$,

$$U_{i,j} = \{x \in \Gamma\colon\ \delta i \leqslant x_d < \delta(i+1), [x_{d+1},\dots,x_n] = j\}$$

for any $i \in [m']$ and $j \in [m_{d+1}] \times \cdots \times [m_n]$, and

$$\mathscr{U}_\Gamma = \{U_{i,j}\colon\ i \in [m'], j \in [m_{d+1}] \times \cdots \times [m_n]\}.$$

Then, $\mathscr{U}_\Gamma$ is a partition of Γ such that $|U_{i,j}| = \mathrm{O}(\log N)$ for any $i \in [m']$ and $j \in [m_{d+1}] \times \cdots \times [m_n]$, and $|\mathscr{U}_\Gamma| = \mathrm{O}(N/\log N)$.

Theorem 2. *If Γ is a finite abelian group represented as $[m_1] \times [m_2] \times \cdots \times [m_n]$ then $C^*(\Gamma,S)[\mathscr{U}_\Gamma]$ is an RFT graph for $C(\Gamma,S)$ with $\mathrm{O}(|E(C(\Gamma,S))| \log N)$ edges.*

Proof. By Theorem 1 and Lemma 6, it suffices to prove that $\lambda(C(\Gamma,S),\mathscr{U}_\Gamma) = \mathrm{O}(|S|N/\log N)$.

Fix $i \in [m']$, $j = [j_{d+1},j_{d+2},\dots,j_n] \in [m_{d+1}] \times \cdots \times [m_n]$, and $s = [s_1,s_2,\dots,s_n] \in S$. If $x = [x_1,x_2,\dots,x_n] \in U_{i,j}$ then $\delta i \leqslant x_d < \delta(i+1)$ and $x_k = j_k (d+1 \leqslant k \leqslant n)$. Thus,

$$\begin{aligned} xs &= [x_1 \oplus_{m_1} s_1, x_2 \oplus_{m_2} s_2, \dots, x_n \oplus_{m_n} s_n] \\ &= [x_1 \oplus_{m_1} s_1, \dots, j_{d+1} \oplus_{m_{d+1}} s_{d+1}, \dots, j_n \oplus_{m_n} s_n]. \end{aligned}$$

Let i' be an integer such that $\delta i' \leqslant \delta i \oplus_{m_d} s_d < \delta(i'+1)$. Since $x_d + s_d = \delta i + s_d + (x_d - \delta i)$ and $0 \leqslant x_d - \delta i < \delta$, we conclude that $xs \in U_{i',j'} \cup U_{i' \oplus_{m'} 1, j'}$ if $\delta \mid m_d$, and $xs \in U_{i',j'} \cup U_{i' \oplus_{m'} 1, j'} \cup U_{i' \oplus_{m'} 2, j}$ otherwise, where $j' = [j_{d+1} \oplus_{m_{d+1}} s_{d+1}, \ldots, j_n \oplus_{m_n} s_n]$. Thus any vertex adjacent with a vertex in $U_{i,j}$ is contained in $U_{i',j'} \cup U_{i' \oplus_{m'} 1, j'} \cup U_{i' \oplus_{m'} 2, j'}$. It follows that

$$\lambda(C(\Gamma,S), \mathscr{U}_\Gamma) \leqslant \frac{1}{2} 3|S|\,|\mathscr{U}_\Gamma| = \mathrm{O}\left(\frac{|S|\,N}{\log N}\right). \qquad \square$$

From Theorems I and 2, we have the following theorem.

Theorem 3. *If Γ is a finite abelian group then we can construct an RFT graph for $C(\Gamma,S)$ with $\mathrm{O}(|E(C(\Gamma,S))| \log N)$ edges.*

3.2.1. RFT graphs for hypercubes and circulant graphs

If $\Gamma = [2] \times \cdots \times [2]$ and $S = \{[1,0,\ldots,0],[0,1,0,\ldots,0],\ldots,[0,\ldots,0,1]\}$ then $C(\Gamma,S)$ is a hypercube. If $\Gamma = [N]$ then $C(\Gamma,S)$ is a circulant graph. Hence, we have the following corollary from Theorem 2.

Corollary 2. *If G is an N-vertex hypercube or circulant graph then we can construct an RFT graph for G with $\mathrm{O}(|E(G)| \log N)$ edges.*

3.3. RFT graphs for Cayley graphs

Let Γ be a group. For any $S, T \subseteq \Gamma$, let $ST = \{xy\colon x \in S, y \in T\}$. If $T = \{y\}$ then we denote ST by Sy. The following theorem is well-known. See, for example, [1] for the proof.

Theorem II. *Let Γ be a group, and let Γ' be a subgroup of Γ. Then, there exists some $\{z_0,\ldots,z_{l-1}\} \subseteq \Gamma$ such that $\{\Gamma' z_0,\ldots,\Gamma' z_{l-1}\}$ is a partition of Γ.*

Let Γ be a group, and let be Γ' be a subgroup with $|\Gamma'| = \Theta(\log N)$. Let $\{z_0,\ldots,z_{l-1}\}$ be a subset of Γ as in Theorem II. Then, $\mathscr{V}_\Gamma = \{\Gamma' z_0,\ldots,\Gamma' z_{l-1}\}$ is a partition of Γ such that $|\Gamma' z_i| = \Theta(\log N)$ for any $i \in [l]$ and $|\mathscr{V}_\Gamma| = \Theta(N/\log N)$.

Theorem 4. *If Γ is a group, and Γ' is a subgroup with $|\Gamma'| = \Theta(\log N)$ then $C^*(\Gamma,S)[\mathscr{V}_\Gamma]$ is an RFT graph for $C(\Gamma,S)$ with $\mathrm{O}(|E(C(\Gamma,S))| \log N)$ edges.*

Proof. By Theorem 1 and Lemma 6, it suffices to prove that $\lambda(C(\Gamma,S), \mathscr{U}_\Gamma) = \mathrm{O}(|S|N/\log N)$.

Notice that if $x \in \Gamma' z_i$ then $\Gamma' x = \Gamma' z_i$. Thus, if $x, y \in \Gamma' z_i$ for some $i \in [l]$ then $xs, ys \in \Gamma' z_j$ for some $j \in [l]$. Thus any vertex adjacent with a vertex in $\Gamma' z_i$ is contained in $\Gamma' z_j$. It follows that

$$\lambda(C(\Gamma,S), \mathscr{V}_\Gamma) \leqslant \frac{1}{2}|S|\frac{N}{|\Gamma'|} = \mathrm{O}\left(\frac{|S|\,N}{\log N}\right). \qquad \square$$

3.3.1. RFT graphs for CCC's and Butterfly-Like graphs

It is proved in [4] that a CCC is a Cayley graph. For any positive integer n, A_n is the $n \times n$ matrix defined as follows:

$$A_n = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

For any positive integer n, let $\mathscr{C}_n$ be the set $[2]^n \times [n]$ with a binary operation defined as follows: For any $[\boldsymbol{x}, i], [\boldsymbol{y}, j] \in \mathscr{C}_n$,

$$[\boldsymbol{x}, i][\boldsymbol{y}, j] = [\boldsymbol{x} \oplus_2 \boldsymbol{y} A_n^i, i \oplus_n j],$$

where the first addition is bit-wise. It is easy to see that $\mathscr{C}_n$ is a group. Let

$$D_n = \{[[0,0,\ldots,0],1],\ [[0,0,\ldots,0],n-1],\ [[1,0,\ldots,0],0]\}, \quad \text{and}$$

$$B_n = \{[[0,0,\ldots,0],1],\ [[0,0,\ldots,0],n-1],\ [[1,0,\ldots,0],1],\ [[0,\ldots,0,1],n-1]\}.$$

Then, $C(\mathscr{C}_n, D_n)$ and $C(\mathscr{C}_n, B_n)$ are the n-dimensional CCC and wrapped butterfly, respectively. Define that

$$\mathscr{C}'_n = \{[\boldsymbol{x}, 0] \in \mathscr{C}_n \colon \forall k \in [n - \lceil \log n \rceil](x_k = 0)\},$$

where $\boldsymbol{x} = [x_0, x_1, \ldots, x_{n-1}] \in [2]^n$.

Lemma 7. *$\mathscr{C}'_n$ is a subgroup of $\mathscr{C}_n$ with $|\mathscr{C}'_n| = \Theta(\log N)$, where $N = |\mathscr{C}_n| = n2^n$.*

Proof. Since $[\boldsymbol{x},0][\boldsymbol{y},0] = [\boldsymbol{x} \oplus_2 \boldsymbol{y}, 0] \in \mathscr{C}'_n$ and $[\boldsymbol{x},0][\boldsymbol{x},0] = [\mathbf{0},0]$ for any $[\boldsymbol{x},0],[\boldsymbol{y},0] \in \mathscr{C}'_n$, $\mathscr{C}'_n$ is a subgroup of $\mathscr{C}_n$. Since $|\mathscr{C}'_n| = 2^{\lceil \log n \rceil}$ and $N = n2^n$, we conclude that $|\mathscr{C}'_n| = \Theta(\log N)$. $\square$

By Theorem 4 and Lemma 7, we obtain the following theorem.

Theorem 5. *If G is an N-vertex CCC or wrapped butterfly then we can construct an RFT graph for G with $\mathrm{O}(|E(G)| \log N)$ edges.*

The following is immediate.

Corollary 3. *If G is an N-vertex butterfly or Beneš network then we can construct an RFT graph for G with $\mathrm{O}(|E(G)| \log N)$ edges.*

3.3.2. RFT graphs for Star and Pancake graphs

Let $[n]^+ = \{1, 2, \ldots, n\}$ for any positive integer n. Let $\mathscr{S}_n$ denote the symmetric group on $[n]^+$, that is the group of the permutations on $[n]^+$. For any integer k, $2 \leqslant k \leqslant n$, let τ_k and π_k be permutations on $[n]^+$ as follows:

$$\tau_k(i) = \begin{cases} k+1-i & \text{if } i=1 \text{ or } i=k, \\ i & \text{otherwise,} \end{cases}$$

$$\pi_k(i) = \begin{cases} k+1-i & \text{if } i \in [k]^+, \\ i & \text{otherwise.} \end{cases}$$

Let

$$S_n = \{\tau_k \colon 2 \leqslant k \leqslant n\} \quad \text{and} \quad P_n = \{\pi_k \colon 2 \leqslant k \leqslant n\}.$$

Then, $C(\mathscr{S}_n, S_n)$ and $C(\mathscr{S}_n, P_n)$ are the star graph and pancake graph on $n!$ vertices, respectively. Let $k = \lceil \log n + \log\log n \rceil$. For any $\boldsymbol{x} = [x_1, x_2, \ldots, x_k] \in [2]^k$,

$$v_{\boldsymbol{x}}(i) = \begin{cases} 2j - (r \oplus_2 x_j) & \text{if } i \in [2k]^+, \text{ and} \\ i & \text{otherwise,} \end{cases}$$

where $j = \lceil i/2 \rceil$ and $r = 2j - i$. Define that

$$\mathscr{S}'_n = \{v_{\boldsymbol{x}} \colon \boldsymbol{x} \in [2]^k\}.$$

Lemma 8. *$\mathscr{S}'_n$ is a subgroup of $\mathscr{S}_n$ with $|\mathscr{S}'_n| = \Theta(\log N)$, where $N = |\mathscr{S}_n| = n!$.*

Proof. Since $v_{\boldsymbol{x}} v_{\boldsymbol{y}} = v_{\boldsymbol{x} \oplus_2 \boldsymbol{y}}$ and $v_{\boldsymbol{x}}^{-1} = v_{\boldsymbol{x}}$ for any $\boldsymbol{x}, \boldsymbol{y} \in [2]^k$, $\mathscr{S}'_n$ is a subgroup of $\mathscr{S}_n$. Since $|\mathscr{S}'_n| = 2^{\lceil \log n + \log\log n \rceil}$ and $N = n!$, we conclude that $|\mathscr{S}'_n| = \Theta(\log N)$. $\square$

By Theorem 4 and Lemma 8, we obtain the following theorem.

Theorem 6. *If G is an N-vertex star graph or pancake graph then we can construct an RFT graph for G with $\mathrm{O}(|E(G)| \log N)$ edges.*

4. RFT graphs for shuffle-exchange and de Bruijn graphs

For any $v = [v_1, v_2, \ldots, v_n] \in [2]^n$, let

$$\sigma(v) = [v_2, \ldots, v_n, v_1], \quad \chi(v) = [v_1, \ldots, v_{n-1}, \overline{v_n}], \quad \text{and} \quad \rho_i(v) = [v_1, \ldots, v_i],$$

where $\overline{v_n}$ denotes the complement of v_n, that is $\overline{v_n} = 1$ if $v_n = 0$, and $\overline{v_n} = 0$ otherwise. Notice that v is the composite of χ and σ, that is, $v = \chi \circ \sigma$.

The n-dimensional de Bruijn graph $dB(n)$ is the graph defined as follows:

$$V(dB(n)) = [2]^n;$$

$$E(dB(n)) = \{(u,v) \colon v = \sigma(u) \text{ or } u = \sigma(v)\}$$
$$\cup \{(u,v) \colon v = \chi(\sigma(u)) \text{ or } u = \chi(\sigma(v))\}.$$

It is easy to see that $|V(dB(n))| = N$ and $|E(dB(n))| = 2N$, where $N = 2^n$.

The n-dimensional shuffle-exchange graph $SE(n)$ is the graph defined as follows:

$$V(SE(n)) = [2]^n;$$

$$E(SE(n)) = \{(u,v)\colon\ v = \sigma(u) \text{ or } u = \sigma(v)\} \cup \{(u,v)\colon\ v = \chi(u)\}.$$

Let

$$V_x = \{v \in [2]^n\colon\ \rho_{n-\lceil \log n \rceil}(v) = x\}$$

for any $x \in [2]^{n-\lceil \log n \rceil}$ and let

$$\mathscr{V}_n = \{V_x\colon\ x \in [2]^{n-\lceil \log n \rceil}\}.$$

Then, $\mathscr{V}_n$ is a partition of $[2]^n$ such that $|V_x| \leqslant 2\log N$ for any $x \in [2]^{n-\lceil \log n \rceil}$ and $|\mathscr{V}_n| \leqslant N/\log N$, where $N = |[2]^n| = 2^n$.

Theorem 7. *$dB^*(n)[\mathscr{V}_n]$ is an RFT graph for $dB(n)$ with $\mathrm{O}(|E(dB(n))| \log N)$ edges.*

Proof. By Theorem 1, it suffices to prove that $\lambda(dB(n), \mathscr{V}_n) = \mathrm{O}(N/\log N)$.

Consider any edge $(u,v) \in E(dB(n))$. Assume without loss of generality that $v=\sigma(u)$ or $v=\chi(\sigma(u))$. Let $x=\rho_{n-\lceil \log n \rceil}(u)$ and $y=\rho_{n-\lceil \log n \rceil}(v)$. It is easy to see that $y=\sigma(x)$ or $y = \chi(\sigma(x))$. Thus,

$$\Lambda(dB(n), \mathscr{V}_n) \subseteq \{(x,y)\colon\ y = \sigma(x) \text{ or } x = \sigma(y)\}$$
$$\cup \{(x,y)\colon\ y = \chi(\sigma(x)) \text{ or } x = \chi(\sigma(y))\},$$

and we have

$$\lambda(dB(n), \mathscr{V}_n) \leqslant 2^{n-\lceil \log n \rceil + 1} \leqslant \frac{2N}{\log N} = \mathrm{O}\left(\frac{N}{\log N}\right). \qquad \square$$

The following can be proved similarly.

Theorem 8. *$SE^*(n)[\mathscr{V}_n]$ is an RFT graph for $SE(n)$ with $\mathrm{O}(|E(SE(n))| \log N)$ edges.*

5. RFT graphs for partial *k*-trees

5.1. Partial *k*-trees

A tree decomposition of a graph G is a pair (T,χ), where T is a tree and $\chi = \{X_t \subseteq V(G)\colon\ t \in V(T)\}$ is a family of subsets of $V(G)$, satisfying the following three conditions:

(1) $V(G) = \bigcup_{t \in V(T)} X_t$;
(2) for every $(u,v) \in E(G)$, there exists $t \in V(T)$ such that $u,v \in X_t$;
(3) for every $r,s,t \in V(T)$, if s is on the path between r and t in T then $X_r \cap X_t \subseteq X_s$.

The width of (T,χ) is $\max\{|X_t|-1\colon t\in V(T)\}$. The treewidth of G is the minimum width over all possible tree decompositions of G. A simple graph of treewidth at most k is called a partial k-tree. A tree is a partial 1-tree. It is not difficult to see that if G is a connected partial k-tree, $N-1\leqslant |E(G)|\leqslant kN$.

5.2. RFT graphs for partial k-trees

We assume in this section that k is a fixed positive integer. Let G be a connected partial k-tree with N vertices, and let (T,χ) be a tree decomposition of G with width at most k, where $\chi=\{X_t\subseteq V(G)\colon t\in V(T)\}$, and T is considered as a rooted tree with root r. For any $t\in V(T)$, $T(t)$ is a subtree of T rooted at t and induced by the descendants of t in T. $\mathrm{child}_T(t)$ is the set of children of t in T. Define that $\Phi_T(t)=\bigcup_{s\in V(T(t))} X_s$, and $\Psi_T(t)=\Phi_T(t)-X_p$, where p is the parent of t in T. Notice that $\Psi_T(r)=V(G)$. We need a few preliminary lemmas.

Lemma 9. *If $t\in V(T)$ and $s,s'\in\mathrm{child}_T(t)$ then $\Psi_T(s)\cap\Psi_T(s')=\emptyset$.*

Proof. Assume contrary that $\Psi_T(s)\cap\Psi_T(s')\neq\emptyset$ and $x\in\Psi_T(s)\cap\Psi_T(s')$. Since t is on the path between s and s' in T, $x\in X_t$ by the definition of the tree decomposition. Since t is the parent of s and s', we conclude that $x\notin\Psi_T(s)\cup\Psi_T(s')$, a contradiction. □

For any $t\in V(T)$, define that

$$\rho_T(t)=\max\{|\Psi_T(s)|\colon s\in\mathrm{child}_T(t)\}.$$

If t is a leaf then we define that $\rho_T(t)=0$.

Lemma 10. *Let $h\geqslant k+1$ be an integer, and let t be a vertex of T with $|\Psi_T(t)|\geqslant 2h+1$. If $\rho_T(t)\leqslant 2h$ then there exists some $S\subseteq\mathrm{child}_T(t)$ such that*

$$h+1\leqslant\left|\bigcup_{s\in S}\Psi_T(s)\right|\leqslant 2h.$$

Proof. Let $\mathrm{child}_T(t)=\{s_0,s_1,\ldots,s_{m-1}\}$, and d be a minimal integer satisfying that $|\bigcup_{i\in[d]}\Psi_T(s_i)|\geqslant h+1$. Since $|\bigcup_{i\in[m]}\Psi_T(s_i)|=|\Psi_T(t)-X_t|\geqslant(2h+1)-(k+1)\geqslant h+1$, there exists such d. If $|\bigcup_{i\in[d]}\Psi_T(s_i)|\leqslant 2h$ then $\{s_0,\ldots,s_{d-1}\}$ is a desired set. Suppose that $|\bigcup_{i\in[d]}\Psi_T(s_i)|\geqslant 2h+1$. Since s_i is a child of t in T for any i, we obtain from Lemma 9 that $|\Psi_T(s_{d-1})|=|\bigcup_{i\in[d]}\Psi_T(s_i)|-|\bigcup_{i\in[d-1]}\Psi_T(s_i)|$. Since $|\bigcup_{i\in[d-1]}\Psi_T(s_i)|\leqslant h$ by the minimality of d, we have $|\Psi_T(s_{d-1})|\geqslant(2h+1)-h=h+1$. Since $|\Psi_T(s_{d-1})|\leqslant\rho_T(t)\leqslant 2h$, $\{s_{d-1}\}$ is a desired set. □

Lemma 11. *Let $h\geqslant k+1$ be an integer, and $|V(G)|\geqslant 2h+1$. Then, there exist some $t\in V(T)$ and $S\subseteq\mathrm{child}_T(t)$ such that*

$$h+1\leqslant\left|\bigcup_{s\in S}\Psi_T(s)\right|\leqslant 2h.$$

```
Input: Graph G and Tree-Decomposition (T, X) of G;
Output: Y = {Y_0, Y_1, ..., Y_{l-1}} and P = {p_0, p_1, ..., p_{l-2}};

begin
   G' := G; T' := T; X' := X; t := r; h := ⌈log |V(G)|⌉; i = 0;
   while |V(G')| ≥ 2h + 1 do begin
      while |Ψ_T'(t)| ≥ 2h + 1 and ρ_T(t) ≥ 2h + 1 do begin
         Select a child s of t such that |Ψ_T'(s)| = ρ_T'(t); t := s
      end;
      Find S ⊂ child_T'(t) such that h + 1 ≤ |∪_{s∈S} Ψ_T'(s)| ≤ 2h;
      Y_i := ∪_{s∈S} V(T'(s)); p_i := t;
      G' := G' − ∪_{s∈S} Ψ_T'(s); T' := T' − Y_i;
      X' := {X_t ∈ X : t ∈ V(T')}; i := i + 1
   end;
   Y_i := V(T')
end
```

Fig. 1. Algorithm 1.

Proof. We recursively define a sequence of vertices in T as follows: (1) $t_1 = r$; (2) If t_i is not a leaf in T then t_{i+1} is a vertex $s \in \mathrm{child}_T(t_i)$ such that $|\Psi_T(s)| = \rho_T(t_i)$. Notice that $|\Psi_T(t_i)| \geqslant |\Psi_T(t_{i+1})|$ for any i, $|\Psi_T(t_1)| = |\Psi_T(r)| = |V(G)| \geqslant 2h + 1$, and if t_i is a leaf of T then $|\Psi_T(t_i)| \leqslant k + 1 \leqslant h$. Thus, there exists a positive integer j such that $|\Psi_T(t_j)| \geqslant 2h + 1$ and $|\Psi_T(t_{j+1})| \leqslant 2h$. Since $\rho_T(t_j) = |\Psi_T(t_{j+1})| \leqslant 2h$, it follows from Lemma 10 that there exists some $S \subseteq \mathrm{child}_T(t_j)$ such that $h + 1 \leqslant |\bigcup_{s\in S} \Psi_T(s)| \leqslant 2h$. □

Now, we are ready to prove a key lemma.

Lemma 12. *There exists a partition* $\mathscr{Y} = \{Y_0, Y_1, \ldots, Y_{l-1}\}$ *of* $V(T)$ *that satisfies the following four conditions*:

(1) $l = \mathrm{O}(N/\log N)$;
(2) *For any* $i \in [l-1]$, *there exists a vertex* $p_i \in V(T)$ *such that the parent of each vertex of* Y_i *is contained in* $Y_i \cup \{p_i\}$;
(3) $r \in Y_{l-1}$, *and the parent of each vertex of* $Y_{l-1} - \{r\}$ *is contained in* Y_{l-1}.
(4) $\mathscr{V} = \{\bigcup_{t\in Y_i} X_t - X_{p_i}\colon i \in [l-1]\} \cup \{\bigcup_{t\in Y_{l-1}} X_t\}$ *is a partition of* $V(G)$ *such that the size of each block is* $\mathrm{O}(\log N)$.

Proof. We define $\mathscr{Y} = \{Y_0, Y_1, \ldots, Y_{l-1}\}$ as the output of Algorithm 1 shown in Fig. 1.It should be noted that the algorithm is based on the previous lemmas. We show that $\mathscr{Y}$ is a desired partition by a series of claims. The first four claims are rather obvious.

Claim 1. $\mathscr{Y} = \{Y_0, Y_1, \ldots, Y_{l-1}\}$ *is a partition of* $V(T)$.

Claim 2. $l = \mathrm{O}(N/\log N)$.

Claim 3. *The parent of each vertex of* Y_i *is contained in* $Y_i \cup \{p_i\}$. $(i \in [l-1])$.

Claim 4. $r \in Y_{l-1}$, *and the parent of each vertex of* $Y_{l-1} - \{r\}$ *is contained in* Y_{l-1}.

Claim 5. *Let* $0 \leqslant i < j \leqslant l-1$. *For any* $s \in Y_i$ *and* $t \in Y_j$, p_i *is on the path between* s *and* t *in* T.

Proof. Assume contrary that p_i is not on the path between s and t in T. Then, there exists a child s' of p_i such that $s,\ t \in V(T(s'))$ since s is a descendant of p_i. Since $s' \in Y_i$, we conclude that $j \leqslant i$, a contradiction. □

Claim 6.

$$\mathscr{V} = \left\{ \bigcup_{t \in Y_i} X_t - X_{p_i}\colon\ i \in [l-1] \right\} \cup \left\{ \bigcup_{t \in Y_{l-1}} X_t \right\}$$

is a partition of $V(G)$ *such that the size of each block is* $\mathrm{O}(\log N)$.

Proof. Let

$$V_i = \begin{cases} \displaystyle\bigcup_{t \in Y_i} X_t - X_{p_i} & \text{if } i \in [l-1], \text{ and} \\ \displaystyle\bigcup_{t \in Y_{l-1}} X_t & \text{if } i = l-1. \end{cases}$$

Assume contrary that $(V_0, V_1, \ldots, V_{l-1})$ is not a partition of $V(G)$. Since $\bigcup_{i \in [l]} V_i = V(G)$, $V_i \cap V_j \neq \emptyset$ for some distinct integers i and j. Assume without loss of generality that $i < j$. Then, there exists a vertex $v \in V(G)$ such that $v \in X_s - X_{p_i}$ for some $s \in Y_i$, and $v \in X_t$ for some $t \in Y_j$. Since $i < j$, p_i is on the path between s and t in T by Claim 5. It follows from the definition of tree decomposition that $v \in X_{p_i}$, a contradiction. Thus, $\mathscr{V}$ is a partition of $V(G)$. Moreover, $|V_i| \leqslant 2h = 2\lceil \log N \rceil$ for any $i \in [l]$ by Lemma 11. □

This completes the proof of Lemma 12. □

Theorem 9. *A partial k-tree G with N vertices has an RFT graph with* $\mathrm{O}(|E(G)| \log N)$ *edges.*

Proof. Consider a partition $\mathscr{V}$ of $V(G)$ defined in Lemma 12. For any $i \in [l-1]$, $\Gamma(i) = \{j > i$: there exist $u \in V_i$ and $v \in V_j$ such that $(u,v) \in E(G)\}$. If $j \in \Gamma(i)$, there exists an edge $(u,v) \in E(G)$ such that $u \in X_s - X_{p_i}$ for some $s \in Y_i$, and $v \in X_t$ for some $t \in Y_j$. From the definition of the tree decomposition, there exists some t' such that $u,\ v \in X_{t'}$. Since $u \notin X_{p_i}$, there exists a child s' of p_i such that $s,\ t' \in V(T(s'))$, and thus $s' \in Y_i$. Since $s' \in Y_i$, $t \in Y_j$, and $i < j$, p_i is on the path between s' and t by Claim 5. Notice that s' is on the path between p_i and t'. Thus, p_i is on the path between t and t' in T. Since $v \in X_t \cap X_{t'}$, $v \in X_{p_i}$ by the definition of tree decomposition. Thus we conclude that $|\Gamma(i)| \leqslant |X_{p_i}| \leqslant k+1$. It follows that

$$\lambda(G,\mathscr{V}) \leqslant l + \sum_{i\in[l-1]} |\Gamma(i)| \leqslant l + \sum_{i\in[l-1]} (k+1)$$

$$= l + (l-1)(k+1) = \mathrm{O}\left(\frac{N}{\log N}\right).$$

Hence, $G^*[\mathscr{V}]$ is an RFT graph for G with $\mathrm{O}(|E(G)|\log N)$ edges by Theorem 1. □

It should be noted that Theorem 9 is a natural generalization of a result for trees shown in [5], since trees are partial 1-trees.

We conclude with a remark on an open problem posed in [5]:

Problem. *Does every graph with N vertices and M edges have an RFT graph with* $\mathrm{O}(N)$ *vertices and* $\mathrm{O}(M\log N)$ *edges*?

The answer to the problem is affirmative if every graph G has a partition $\mathscr{V} = (V_0, V_1, \ldots, V_{k-1})$ of $V(G)$ such that:

- $|V_i| = \mathrm{O}(\log N)$ for $\forall i$,
- $k = \mathrm{O}(N/\log N)$, and
- $\lambda(G,\mathscr{V}) = \mathrm{O}(M/\log N)$.

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