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# On sequential diagnosis of multiprocessor systems

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**Abstract**

This paper considers the problem of sequential fault diagnosis for various multiprocessor systems. We propose a simple sequential diagnosis algorithm and show that the degree of sequential diagnosability of any system with $N$ processors is at least $\Omega(\sqrt{N})$. We also show upper bounds for various networks. These are the first nontrivial upper bounds for the degree of sequential diagnosability, to the best of our knowledge. Our upper bounds are proved in a unified manner, which is based on the very definition of sequential diagnosability. We show that a $d$-dimensional grid and torus with $N$ vertices are sequentially $\mathrm{O}(N^{d/(d+1)})$-diagnosable, and an $N$-vertex $k$-ary tree is $\mathrm{O}(\sqrt{kN})$-diagnosable. Moreover, we prove that the degree of sequential diagnosability of an $N$-vertex hypercube is at least $\Omega(N/\sqrt{\log N})$ and at most $\mathrm{O}(N \log \log N/\sqrt{\log N})$, and those of an $N$-vertex CCC, shuffle-exchange graph, and de Bruijn graph are $\Theta(N/\log N)$.


*Keywords:* System diagnosis; Degree of sequential diagnosability; Grid; Torus; $k$-ary tree; Hypercube; Cube-connected cycles; Shuffle-exchange network; de Bruijn network

## 1. Introduction

The system diagnosis has been extensively studied in the literature in connection with fault-tolerant multiprocessor computer systems. An original graph-theoretical model for system diagnosis was introduced in a classic paper by Preparata et al. [10]. In this model, the testing assignment is represented by a digraph (directed graph) associated with the interconnection graph of the system. The model assumes that the processors can test each

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other along available communication links. A testing processor evaluates a tested processor as fault-free or faulty. A syndrome is a collection of test results. The model also assumes that the number of faulty processors is bounded.

Two strategies for the diagnosis were introduced and discussed in [10]. A system is said to be one-step $t$-diagnosable if all faulty processors can be identified uniquely from any syndrome provided that the number of faulty processors does not exceed $t$. A system is said to be sequentially $t$-diagnosable if at least one faulty processors can be identified from any syndrome provided that the number of faulty processors does not exceed $t$. The degree of one-step (sequential) diagnosability of a system is the maximal $t$ such that the system is one-step (sequentially) $t$-diagnosable. A characterization of one-step $t$-diagnosable systems by Hakimi and Amin [3] implies that the degree of one-step diagnosability of any system is bounded by the minimum degree of a vertex in its interconnection graph. On the other hand, it is known that the degree of sequential diagnosability of any system with $N$ processors is at least $\Omega(\sqrt[3]{N})$ [6]. Unfortunately, computing the degree of sequential diagnosability of a system is co-NP hard as proved by Raghavan and Tripathi [11].

The grid, hypercube, tree, cube-connected cycles (CCC), shuffle-exchange graph, and de Bruijn graph are popular interconnection graphs for multiprocessor computer systems. The sequential diagnosis for hypercube was first considered by Kavianpour and Kim [4]. They proved that the degree of sequential diagnosability for an $N$-vertex hypercube is at least $\Omega(\sqrt{N \log N})$ [4]. Khanna and Fuchs also showed the same lower bound by giving a linear time algorithm for sequential diagnosis for hypercube [5]. Moreover, they showed in [6] that the degree of sequential diagnosability for an $N$-vertex hypercube is at least $\Omega(N \log\log N/\log N)$. In the same paper [6], they proved that the degree of sequential diagnosability for a $d$-dimensional grid with $N$ vertices is at least $\Omega(N^{d/(d+1)})$. In addition, it is shown in [6] that the degree of sequential diagnosability of any system with $N$ processors is at least $\Omega(\sqrt{N/\Delta})$ and $\Omega(\Delta)$, where $\Delta$ is the maximum degree of a vertex in its interconnection graph. The former lower bound is based on a sequential diagnosis algorithm called PARTITION, while the latter is based on another algorithm called MAX. It follows that the degree of sequential diagnosability of a $k$-ary tree with $N$ vertices is $\Omega(\sqrt{N/k})$ and those of an $N$-vertex CCC, shuffle-exchange graph, and de Bruijn graph are $\Omega(\sqrt{N})$. From lower bounds of $\Omega(\sqrt{N/\Delta})$ and $\Omega(\Delta)$ mentioned above, we can derive a general lower bound of $\Omega(\sqrt[3]{N})$, which is independent of $\Delta$ [6]. However, we know no graph $G$ with $N$ vertices such that the degree of sequential diagnosability is $o(\sqrt{N})$.

This paper first shows that this is indeed the case by proving that the degree of sequential diagnosability of any system with $N$ processors is $\Omega(\sqrt{N})$. Our lower bound is based on a sequential diagnosis algorithm called HYBRID, which is a natural common generalization of algorithms PARTITION and MAX proposed in [6]. We next show upper bounds for various networks. These are the first nontrivial upper bounds for the degree of sequential diagnosability, to the best of our knowledge. Our upper bounds are proved in a unified manner, which is based on the very definition of sequential diagnosability. We show that a $d$-dimensional grid and torus with $N$ vertices are sequentially $O(N^{d/(d+1)})$-diagnosable, and an $N$-vertex $k$-ary tree is $O(\sqrt{kN})$-diagnosable. Finally, we prove that the degree of sequential diagnosability of an $N$-vertex hypercube is at least $\Omega(N/\sqrt{\log N})$ and at most $O(N \log\log N/\sqrt{\log N})$, and those of an $N$-vertex CCC, shuffle-exchange graph, and de Bruijn graph are $\Theta(N/\log N)$.

Preliminary versions of the paper appeared in [8,9,13,14].

## 2. Sequential diagnosis

The interconnection network of a multiprocessor computer system is modeled by a graph, called an interconnection graph, with the processors represented by the vertices of the graph and the communication links by the edges. The testing assignment in the system is modeled by a digraph, called a testing digraph, with the processors represented by the vertices of the digraph and the test by the arcs (directed edges). If $\langle x, y\rangle$ is an arc of the testing digraph then the processor $x$ tests processor $y$. A test is performed along an edge of the interconnection graph.

We denote the vertex set and the edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. We also denote the vertex set and the arc set of a digraph $D$ by $V(D)$ and $A(D)$, respectively. The associated digraph $\widehat{G}$ of a graph $G$ is the digraph obtained when each edge $e$ of $G$ is replaced with two oppositely oriented arcs with the same ends as $e$.

Let $D$ be a testing digraph of a system. A syndrome for $D$ is a mapping $\sigma : A(D) \to \{0, 1\}$ defined as follows:

$$\sigma\langle x, y\rangle = \begin{cases} 0 & \text{if } x \text{ tests } y \text{ with outcome pass,} \\ 1 & \text{if } x \text{ tests } y \text{ with outcome fail,} \end{cases}$$

where we denote $\sigma(\langle x, y\rangle)$ simply by $\sigma\langle x, y\rangle$. The outcome of the test is considered reliable if and only if $x$ is fault-free. A set $F \subseteq V(D)$ is said to be a consistent fault set for a syndrome $\sigma$ if neither (i) nor (ii) below holds:

(i) $\sigma\langle x, y\rangle = 0$ where $x \in V(D) - F$ and $y \in F$,
(ii) $\sigma\langle x, y\rangle = 1$ where $x, y \in V(D) - F$.

For any syndrome $\sigma$ for $D$ and positive integer $t$, define

$$\mathscr{F}(\sigma, t) = \{F : F \subseteq V(D) \text{ is a consistent fault set for } \sigma \text{ and } |F| \leqslant t\},$$
$$\mathscr{S}_D(t) = \{\sigma : \mathscr{F}(\sigma, t) \neq \emptyset\}.$$

$D$ is said to be sequentially $t$-diagnosable if

$$|\mathscr{F}(\sigma, t)| = 1 \text{ or } \bigcap\{F : F \in \mathscr{F}(\sigma, t)\} \neq \emptyset$$

for any syndrome $\sigma \in \mathscr{S}_D(t)$. The degree of sequential diagnosability for $D$, denoted by $\delta(D)$, is the largest integer $t$ for which $D$ is sequentially $t$-diagnosable.

## 3. Algorithm HYBRID

In this section, we propose a linear-time sequential diagnosis algorithm for strongly-connected testing digraphs.

Let $D$ be a testing digraph of a system, and let $\sigma$ be a syndrome for $D$. $D_{\sigma=0}$ is the digraph defined as follows:

$$V(D_{\sigma=0}) = V(D); \quad A(D_{\sigma=0}) = \{\langle x, y\rangle \in A(D):\ \sigma\langle x, y\rangle = 0\}.$$

**Lemma 1.** *Let $F$ be a consistent fault set for $\sigma$ and let $X$ be a strongly-connected component of $D_{\sigma=0}$. Then, either $X \cap F = \emptyset$ or $X \subseteq F$.*

**Proof.** Assume that $X \not\subseteq F$. Then, there exists some $u \in X - F$. Consider any $v \in X$. Since $X$ is a strongly-connected component of $D_{\sigma=0}$, there exists a dipath from $u$ to $v$ in $D_{\sigma=0}$. Notice that by the definition of syndrome, if $x \notin F$ and $\langle x, y\rangle \in A(D_{\sigma=0})$ then $y \notin F$. Thus, we conclude that $v \notin F$, and hence $X \cap F = \emptyset$. □

$\overline{D}(\sigma)$ is the graph defined as follows:

$$V(\overline{D}(\sigma)) = \{X:\ X \text{ is a strongly-connected component of } D_{\sigma=0}\};$$

$$E(\overline{D}(\sigma)) = \{(X, Y):\ \sigma\langle x, y\rangle = 1 \text{ or } \sigma\langle y, x\rangle = 1 \text{ for some } x \in X \text{ and } y \in Y\}.$$

**Lemma 2.** *For any $(X, Y) \in E(\overline{D}(\sigma))$, $X \subseteq F$ or $Y \subseteq F$.*

**Proof.** Assume $X \not\subseteq F$. Then, $X \cap F = \emptyset$ by Lemma 1. Notice that $\sigma\langle x, y\rangle = 1$ or $\sigma\langle y, x\rangle = 1$ for some $x \in X$ and $y \in Y$ by the definition of $\overline{D}(\sigma)$. Since $x \notin F$, we conclude that $y \in F$, and hence $Y \subseteq F$ by Lemma 1. □

For any $X \in V(\overline{D}(\sigma))$, define that

$$\mathcal{N}_\sigma(X) = \bigcup\{Y:\ (X, Y) \in E(\overline{D}(\sigma))\}.$$

**Lemma 3.** *Let $t$ be a positive integer, $\sigma \in \mathcal{S}_D(t)$, and $F \in \mathcal{F}(\sigma, t)$. Then, if $|\mathcal{N}_\sigma(X)| \geqslant t+1$ for some $X \in V(\overline{D}(\sigma))$, $X \subseteq F$.*

**Proof.** If $(X, X) \in E(\overline{D}(\sigma))$ then $X \subseteq F$ by Lemma 2.

Consider the case when $(X, X) \notin E(\overline{D}(\sigma))$, and assume contrary that $X \not\subseteq F$. By Lemma 2, $Y \subseteq F$ for any $(X, Y) \in E(D(\sigma))$. Hence, we conclude that $\mathcal{N}_\sigma(X) \subseteq F$, which is contradicting to the fact that $|F| \leqslant t$. Hence, $X \subseteq F$. □

Define that

$$n_\sigma(X) = \begin{cases} |\mathcal{N}_\sigma(X)| & \text{if } \mathcal{N}_\sigma(X) \neq \emptyset, \\ |X| & \text{otherwise,} \end{cases}$$

and

$$v_D(t) = \min_{\sigma \in \mathcal{S}_D(t)} \max_{X \in V(\overline{D}(\sigma))} n_\sigma(X).$$

**Theorem 1.** *Let $D$ be a strongly-connected testing digraph. If $v_D(t) \geqslant t+1$ for a positive integer $t$ then $D$ is sequentially $t$-diagnosable.*

**Proof.** Consider any syndrome $\sigma \in \mathscr{S}_D(t)$. By the definition of $v_D(t)$, there exists some $X \in V(\overline{D}(\sigma))$ with $n_\sigma(X) \geqslant t+1$. Let $F \in \mathscr{F}(\sigma, t)$. There are following two cases:

*Case* 1: $\mathscr{N}_\sigma(X) \neq \emptyset$: Then, $n_\sigma(X) = |\mathscr{N}_\sigma(X)| \geqslant t+1$, and so $X \subseteq F$ by Lemma 3.

*Case* 2: $\mathscr{N}_\sigma(X) = \emptyset$: Then, $n_\sigma(X) = |X| \geqslant t+1$, and so $X \cap F = \emptyset$. If $\sigma\langle x, y\rangle = 0$ for every $\langle x, y\rangle \in A(D)$ then $X = V(D)$, and hence $F = X \cap F = \emptyset$. If $\sigma\langle x, y\rangle = 1$ for some $\langle x, y\rangle \in A(D)$ then there exists a dipath $P = \langle v_0, v_1, \ldots, v_k\rangle$ in $D$ such that $v_0 \in X$, $\sigma\langle v_{i-1}, v_i\rangle = 0$ for any $i = 1, 2, \ldots, k-1$, and $\sigma\langle v_{k-1}, v_k\rangle = 1$. Since $v_0 \notin F$, we conclude that $v_k \in F$ by the definition of syndrome.

Hence $D$ is sequentially $t$-diagnosable. $\square$

Now, we are ready to describe a linear time sequential diagnosis algorithm for strongly-connected testing digraphs based on Theorem 1. Fig. 1 shows our algorithm, referred to HYBRID. It is easy to see the correctness of HYBRID from the proof of Theorem 1. Phase 1 is performed in $\mathrm{O}(|A(D)|)$ time, and Phase 2 is performed in $\mathrm{O}(|A(D)|)$ time by using the depth-first search. Thus, we obtain the following:

**Theorem 2.** *Let D be a strongly-connected testing graph, and let t be a positive integer with* $v_D(t) \geqslant t+1$. *Then, HYBRID diagnoses correctly all faulty processors in D in linear time provided that D has at most t faulty processors*.

## 4. General lower bound

Let $G$ be a connected graph and let $\sigma$ be a syndrome for $\widehat{G}$. We denote $\overline{\widehat{G}}(\sigma)$ by $G(\sigma)$ for simplicity.

**Lemma 4.** $G(\sigma)$ *is connected*.

**Proof.** The lemma follows from the fact that if $(x, y) \in E(G)$ then $x, y \in X$, or $x \in X$, $y \in Y$, and $(X, Y) \in E(G(\sigma))$. $\square$

For any $X \in V(G(\sigma))$, define that

$$\Gamma_\sigma(X) = \mathscr{N}_\sigma(X) \cup X.$$

**Lemma 5.** *Let F be a consistent fault set for σ. Then, there exists a partition* $(X_1, X_2, \ldots, X_m)$ of $F$ *such that*

(1) $X_i \in V(G(\sigma))$ *for any positive integer* $i \leqslant m$, *and*
(2) $\Gamma_\sigma(X_i) \cap \bigcup_{j=1}^{i-1} \Gamma_\sigma(X_j) \neq \emptyset$ *for any integer* $2 \leqslant i \leqslant m$,
*where* $m = |\{X \in V(G(\sigma)) : X \subseteq F\}|$.

**Proof.** Let

$$\mathscr{X} = \{X \in V(G(\sigma)) : X \subseteq F\}.$$

```
algorithm HYBRID
  input D: testing digraph;
        t: integer s.t. ν_D(t) ≥ t + 1;
begin
{Phase 1}
  Construct a syndrome σ for D;
  Construct D_{σ=0} and D(σ);
  Compute |X| and |N_σ(X)| for ∀X ∈ V(D(σ));
  if ∃X |N_σ(X)| ≥ t + 1 then begin
    Choose X s.t. |N_σ(X)| ≥ t + 1;
    Replace each processor in X with a spare processor; end
  else
    Choose X with |X| ≥ t + 1;
{Phase 2}
  while X ≠ V(D) do begin
    Select any v ∈ V(D) − X s.t. ⟨u, v⟩ ∈ A(D) for ∃u ∈ X;
    Compute σ⟨u, v⟩;
    Replace v with a spare processor if σ⟨u, v⟩ = 1;
    Add v to X;
  end
end
```

Fig. 1. Algorithm HYBRID.

Notice that $\bigcup_{X\in\mathscr{X}} X = F$ by Lemma 1, and $X \cap X' = \emptyset$ for any distinct $X, X' \in \mathscr{X}$ by the definition of $G(\sigma)$. Thus, in order to prove the lemma, it suffices to label the elements of $\mathscr{X}$ as $X_1,\ldots, X_m$ so that condition 2 is satisfied.

Let $X_1$ be any element of $\mathscr{X}$. Suppose that $X_1, \ldots$, and $X_{i-1}$ $(i \geqslant 2)$ are given. Assume contrary that

$$\Gamma_\sigma(X) \cap \bigcup_{j=1}^{i-1} \Gamma_\sigma(X_j) = \emptyset$$

for any $X \in \mathscr{X} - \{X_1, \ldots, X_{i-1}\}$. Then, the distance of $X$ and $X_j$ in $G(\sigma)$ is at least 3 for any $j$ $(1 \leqslant j \leqslant i-1)$. Since $G(\sigma)$ is connected by Lemma 4, there exists some $(U, V) \in E(G(\sigma))$ with $(U \cup V) \cap F = \emptyset$, which is contradicting to Lemma 2. Thus, we have

$$\Gamma_\sigma(X) \cap \bigcup_{j=1}^{i-1} \Gamma_\sigma(X_j) \neq \emptyset.$$

for some $X \in \mathscr{X} - \{X_1, \ldots, X_{i-1}\}$, and we can select such $X$ as $X_i$. $\square$

The following shows a tradeoff between $t$ and $v_{\widehat{G}}(t)$.

**Lemma 6.** *For any N-vertex connected graph G and any positive integer* $t \leqslant N$,

$$t \cdot v_{\widehat{G}}(t) + 1 \geqslant N.$$

**Proof.** Let $\sigma \in \mathscr{S}_D(t)$ such that

$$\max_{X \in V(G(\sigma))} n_\sigma(X) = v_{\widehat{G}}(t).$$

Then, for any $X \in V(G(\sigma))$, we have

$$|\mathscr{N}_\sigma(X)| \leqslant v_{\widehat{G}}(t) \quad \text{and} \quad |\Gamma_\sigma(X)| \leqslant v_{\widehat{G}}(t) + |X|.$$

Fix $F \in \mathscr{F}(\sigma, t)$, and let $(X_1, X_2, \ldots, X_m)$ be a partition of $F$ satisfying the conditions in Lemma 5. For any positive integer $i \leqslant m$, define that

$$Z_i = \begin{cases} \Gamma_\sigma(X_1) & \text{if } i = 1, \\ \Gamma_\sigma(X_i) - \bigcup_{j=1}^{i-1} \Gamma_\sigma(X_j) & \text{otherwise.} \end{cases}$$

It is easy to see that $(Z_1, Z_2, \ldots, Z_m)$ is a partition of $V(\widehat{G})$. Since

$$|Z_i| \leqslant \begin{cases} v_{\widehat{G}}(t) + |X_1| & \text{if } i = 1, \\ v_{\widehat{G}}(t) + |X_i| - 1 & \text{otherwise,} \end{cases}$$

by Lemma 5 and $|V(\widehat{G})| = N$, we conclude that

$$N = \sum_{i=1}^{m} |Z_i| \leqslant \sum_{i=1}^{m} (v_{\widehat{G}}(t) + |X_i| - 1) + 1 = m(v_{\widehat{G}}(t) - 1) + |F| + 1$$
$$\leqslant t \cdot v_{\widehat{G}}(t) + 1. \qquad \square$$

**Theorem 3.** *For any N-vertex connected graph G,*

$$\delta(\widehat{G}) \geqslant \left\lceil \sqrt{N-1} \right\rceil - 1.$$

**Proof.** Selecting $t = \lceil \sqrt{N-1} \rceil - 1$, we have $v_{\widehat{G}}(t) \geqslant \lceil \sqrt{N-1} \rceil = t + 1$ by Lemma 6. Hence, we have $\delta(\widehat{G}) \geqslant t = \lceil \sqrt{N-1} \rceil - 1$ by Theorem 1. $\square$

The $k$-partition number of a graph $G$, denoted by $\Upsilon_G(k)$, is defined as the largest integer $p$ such that for all $p$-element subsets $S \subseteq V(G)$, the subgraph of $G$ induced by the vertices in $V(G) - S$ has a connected component of size $k$ or larger. The following general theorems are proved in [6].

**Theorem 4** (*Khanna and Fuchs [6]*). *If $\Upsilon_G(t+1) \geqslant t$ for some integer t then $\delta(\widehat{G}) \geqslant t$.*

**Theorem 5** (*Khanna and Fuchs [6]*). $\delta(\widehat{G}) \geqslant \lfloor \Delta(G)/2 \rfloor$, *where $\Delta(G)$ is the maximum vertex degree of G.*

It should be noted that our lower bound in Theorem 3 is an improvement on those in Theorems 4 and 5. Let $T_k^3$ be an $N$-vertex complete $k$-ary tree of height 3. It is easy to see that $\Upsilon_{T_k^3}(k+1) = k^2 \geqslant k$ and $\Upsilon_{T_k^3}(k+2) = k - 1 < k + 1$. It is also easy to see that

$\Delta(T_k^3) = k+1$. Since $N = k^3 + k^2 + k + 1$, the lower bound for $\delta(\widehat{T}_k^3)$ obtained from Theorems 4 and 5 is $k = \Theta(\sqrt[3]{N})$.

Notice also that our lower bound is asymptotically tight in the sense that for any $N$-vertex tree $T$ with bounded degree, $\delta(T) = \mathrm{O}(\sqrt{N})$ as shown in Section 5.2.

## 5. Upper bounds for arrays and trees

Our upper bounds are derived from the following simple observation, which is straightforward from the definition of sequential diagnosability.

**Lemma 7.** *Let D be a testing digraph and t be a positive integer. If there exist a syndrome $\sigma$ for D and a collection $\{F_1, F_2, \dots, F_m\}$ of consistent fault sets for $\sigma$ with $m \geqslant 2$ and $0 < |F_i| \leqslant t$ $(1 \leqslant i \leqslant m)$, such that*

$$\bigcap_{i=1}^{m} F_i = \emptyset$$

*then D is not sequentially t-diagnosable, that is*

$$\delta(D) < t.$$

### 5.1. Grids and Tori

For any positive integer $n$, $[n] = \{0, 1, \dots, n-1\}$. The $d$-dimensional $m$-sided grid, denoted by $R_d(m)$, is defined as follows:

$$V(R_d(m)) = [m]^d, \quad E(R_d(m)) = \left\{ (\boldsymbol{x}, \boldsymbol{y}) : \sum_{i=1}^{d} |x_i - y_i| = 1 \right\},$$

where $\boldsymbol{x} = (x_d, x_{d-1}, \dots, x_1)$ and $\boldsymbol{y} = (y_d, y_{d-1}, \dots, y_1)$. The $d$-dimensional $m$-sided torus, denoted by $D_d(m)$, is defined as follows:

$$V(D_d(m)) = [m]^d;$$

$$E(D_d(m)) = \{(\boldsymbol{x}, \boldsymbol{y}) : (\exists i)[y_i = (x_i \pm 1) \bmod m,\ (\forall j \neq i)[x_j = y_j]]\}.$$

The following lower bound can be found in the literature.

**Theorem 6** (*Khanna and Fuchs [6]*). $\delta(\widehat{R}_d(m)) = \Omega(N^{d/(d+1)})$.

In this subsection, we prove the following upper bound:

**Theorem 7.** $\delta(\widehat{D}_d(m)) = \mathrm{O}(N^{d/(d+1)})$.

Note that $\delta(\widehat{R}_d(m)) \leqslant \delta(\widehat{D}_d(m))$ since $R_d(m)$ is a subgraph of $D_d(m)$. Thus, we have the following two corollaries from Theorems 6 and 7.

**Corollary 1.** $\delta(\widehat{R}_d(m)) = \Theta(N^{d/(d+1)})$.

**Corollary 2.** $\delta(\widehat{D}_d(m)) = \Theta(N^{d/(d+1)})$.

*5.1.1. Proof of Theorem 7*

*5.1.1.1. Partition of $V(\widehat{D}_d(m))$.* Let $\Psi$ be a positive integer. In this subsection, suppose that $\Psi|m$ for simplicity of argument. We can prove the theorem by a similar argument even for the other case. Let $\rho = m/\Psi$. For each $\boldsymbol{t} = (t_d, t_{d-1}, \ldots, t_1) \in [\Psi]^d$, define $P(\boldsymbol{t})$ and $Q(\boldsymbol{t})$ as follows:

$$P(\boldsymbol{t}) = \{\boldsymbol{x} \in V(\widehat{D}_d(m)) : (\forall i)[\lfloor x_i/\rho \rfloor = t_i \text{ and } 1 \leqslant x_i \bmod \rho \leqslant \rho - 2]\},$$

$$Q(\boldsymbol{t}) = \{\boldsymbol{x} \in V(\widehat{D}_d(m)) : (\forall i)[\lfloor x_i/\rho \rfloor = t_i] \text{ and } (\exists j)[x_j \bmod \rho = 0 \text{ or } \rho - 1]\}.$$

It is easy to see that $(P((0,\ldots,0)), \ldots, P((\Psi-1,\ldots,\Psi-1)), Q((0,\ldots,0)), \ldots, Q((\Psi-1,\ldots,\Psi-1)))$ is a partition of $V(\widehat{D}_d(m))$. Let $\mathscr{P} = \bigcup_{\boldsymbol{t}} P(\boldsymbol{t})$ and $\mathscr{Q} = \bigcup_{\boldsymbol{t}} Q(\boldsymbol{t})$.

*5.1.1.2. Syndrome and fault sets.* The syndrome $\sigma_\Psi$ for $\widehat{D}_d(m)$ is defined as follows:

$$\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \begin{cases} 0 & \text{if } \begin{cases} 1. & \boldsymbol{x}, \boldsymbol{y} \in P(\boldsymbol{t}) \text{ for some } \boldsymbol{t}, \text{ or} \\ 2. & \boldsymbol{x}, \boldsymbol{y} \in Q(\boldsymbol{t}) \text{ for some } \boldsymbol{t}, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define $\Psi^d$ fault sets as follows:

$$F(\boldsymbol{t}) = P(\boldsymbol{t}) \cup (\mathscr{Q} - Q(\boldsymbol{t})) \;\; (\boldsymbol{t} \in [\Psi]^d).$$

We prove Theorem 7 by showing the following claims:

**Claim 1.** *For any $\boldsymbol{t} \in [\Psi]^d$, $F(\boldsymbol{t})$ is a consistent fault set for $\sigma_\Psi$.*

**Claim 2.** $\bigcap_{\boldsymbol{t} \in [\Psi]^d} F(\boldsymbol{t}) = \emptyset$.

**Claim 3.** $|F(\boldsymbol{t})| = \mathrm{O}(N^{d/(d+1)})$ *for any* $\boldsymbol{t} \in [\Psi]^d$.

*5.1.1.3. Proof of Claim 1.* We will prove the claim by showing that neither (i) nor (ii) below holds for any $\boldsymbol{t} \in [\Psi]^d$:

(i) $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$ if $\boldsymbol{x} \in V(\widehat{D}_d(m)) - F(\boldsymbol{t})$ and $y \in F(\boldsymbol{t})$,
(ii) $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$ if $\boldsymbol{x}, \boldsymbol{y} \in V(\widehat{D}_d(m)) - F(\boldsymbol{t})$.

Let $F(\boldsymbol{t})$ be a fault set, $\boldsymbol{x} \in V(\widehat{D}_d(m)) - F(\boldsymbol{t})$, and $\langle \boldsymbol{x}, \boldsymbol{y}\rangle \in A(\widehat{D}_d(m))$.

*Case* 1: $\boldsymbol{x} \in P(\boldsymbol{t}')$ for some $\boldsymbol{t}' \neq \boldsymbol{t}$: The vertices adjacent to $\boldsymbol{x}$ are contained in $P(\boldsymbol{t}') \cup Q(\boldsymbol{t}')$.

*Case* 1.1: $\boldsymbol{y} \in F(\boldsymbol{t})$: $\boldsymbol{y} \in Q(\boldsymbol{t}')$ and so $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$.

*Case* 1.2: $\boldsymbol{y} \in V(\widehat{D}_d(m)) - F(\boldsymbol{t})$: $\boldsymbol{y} \in P(\boldsymbol{t}')$ and so $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

*Case* 2: $\boldsymbol{x} \in Q(\boldsymbol{t})$: The vertices adjacent to $\boldsymbol{x}$ are contained in $P(\boldsymbol{t}) \cup \mathscr{Q}$.

*Case* 2.1: $\boldsymbol{y} \in F(\boldsymbol{t})$: $\boldsymbol{y} \notin Q(\boldsymbol{t})$ and so $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$.

*Case* 2.2: $\boldsymbol{y} \in V(\widehat{D}_d(m)) - F(\boldsymbol{t})$: $\boldsymbol{y} \in Q(\boldsymbol{t})$ and so $\sigma_\Psi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

Thus, neither (i) nor (ii) holds for any arc $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$.

*5.1.1.4. Proof of Claim 2.* The claim follows from the fact that $Q(\boldsymbol{t}) \cap F(\boldsymbol{t}) = \emptyset$ for any $\boldsymbol{t}$, and $P(\boldsymbol{t}) \cap F(\boldsymbol{t}') = \emptyset$ for any distinct $\boldsymbol{t}$ and $\boldsymbol{t}'$.

*5.1.1.5. Proof of Claim 3.* Since

$$|P(\boldsymbol{t})| < |\{\boldsymbol{x} \in V(\widehat{D}_d(m)) : (\forall i)[\lfloor x_i/\rho \rfloor = t_i]\} \leqslant \left(\frac{m}{\Psi}\right)^d$$

and

$$\begin{aligned} |\mathcal{Q}| &< |\{\boldsymbol{x} \in V(\widehat{D}_d(m)) : (\exists i)[x_i \bmod \rho = 0 \text{ or } \rho - 1]\} \\ &< \sum_{i=1}^{d} \sum_{j=0}^{\Psi-1} |\{\boldsymbol{x} : x_i = j\rho\}| + \sum_{i=1}^{d} \sum_{j=0}^{\Psi-1} |\{\boldsymbol{x} : x_i = (j+1)\rho - 1\}| \\ &= 2d\Psi m^{d-1}, \end{aligned}$$

we conclude that

$$|F(\boldsymbol{t})| = |P(\boldsymbol{t}) \cup (\mathcal{Q} - Q(\boldsymbol{t}))| < |P(\boldsymbol{t}) \cup \mathcal{Q}| < \left(\frac{m}{\Psi}\right)^d + 2d\Psi m^{d-1}.$$

If we choose $\Psi = \lceil (m/d)^{1/(d+1)} \rceil$, we have

$$|F(\boldsymbol{t})| = \mathrm{O}((dm^d)^{d/(d+1)}) = \mathrm{O}(N^{d/(d+1)}).$$

### 5.2. *k-ary trees*

Let $T$ be a rooted tree with root $r$. For any $v \in V(T)$, the level of $v$, denoted by $l_T(v)$, is defined as the number of edges of the unique path connecting $v$ and $r$. A rooted tree $T$ is said to be of height $h$ if $\max\{l_T(v) : v \in V(T)\} = h$. A vertex $v$ is called an ancestor of a vertex $u$ (and $u$ is called a descendant of $v$) if $v$ is on the unique path in $T$ connecting $r$ and $u$. If $v$ is an ancestor of $u$ and $(u, v) \in E(T)$ then $v$ is the parent of $u$ (and $u$ is a child of $v$). If each vertex of $T$ has at most $k$ children then $T$ is called a $k$-ary tree.

Let $T_{k,N}$ denote an $N$-vertex $k$-ary tree. In this subsection, we prove the following upper bound:

**Theorem 8.** $\delta(\widehat{T}_{k,N}) = \mathrm{O}(\sqrt{kN})$.

The following corollary is a direct consequence of Theorems 3 and 8.

**Corollary 3.** *If k is fixed,* $\delta(\widehat{T}_{k,N}) = \Theta(\sqrt{N})$.

```
algorithm Partition_k_Tree
  input T_{k,N}: N-vertex k-ary tree;
        r: root of T_{k,N};
        Ψ: integer;
  output P(1), ..., P(m): subsets of V(T_{k,N});
         q_1, ..., q_m: vertex of T_{k,N};
begin
  T := T_{k,N}; i := 1;
  while |V(T)| ≥ kΨ + 1 do begin
    r' := r;
    while |V(T(r'))| ≥ kΨ + 1 do r' := child_T(r');
    P(i) := V(T(r')) − {r'}; q_i := r';
    T := T − V(T(r')); i := i + 1;
  end
  P(i) := V(T) − {r}; q_i := r;
end
```

Fig. 2. Algorithm partition_*k*_tree

#### 5.2.1. *Proof of Theorem 8*

*5.2.1.1. Partition of* $V(T_{k,N})$. Let $T$ be a rooted tree. For any $r' \in V(T)$, let $T(r')$ denote the rooted subtree of $T$ with root $r'$ induced by the descendants of $r'$, and let $\mathrm{child}_T(r')$ be a child $x$ of $r'$ such that

$$|V(T(x))| = \max\{|V(T(y))| : y \text{ is a child of } r'\}.$$

Let $P(1), \ldots, P(m), q_1, \ldots, q_m$ be the output of the Partition_*k*_Tree algorithm shown in Fig. 2. Define $\mathscr{P} = \bigcup_i P(i)$ and $\mathscr{Q} = \{q_1, \ldots, q_m\}$. It is easy to see that $(P(1), \ldots, P(m), \mathscr{Q})$ is a partition of $V(T_{k,N})$, and thus a partition of $V(\widehat{T}_{k,N})$.

*5.2.1.2. Syndrome and fault sets.* The syndrome $\sigma_\Psi$ for $\widehat{T}_{k,N}$ is defined as follows:

$$\sigma_\Psi\langle x, y\rangle = \begin{cases} 1 & \text{if } x \in \mathscr{Q} \text{ or } y \in \mathscr{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

We define $m+1$ fault sets as follows: For any integer $i$ $(0 \leqslant i \leqslant m)$

$$F(i) = \begin{cases} \mathscr{Q} & \text{if } i = 0, \\ P(i) \cup (\mathscr{Q} - \{q_i\}) \cup R(i) & \text{otherwise.} \end{cases}$$

where $R(i) = P(j)$ if the parent of $q_i$ is in $P(j)$ for some $j$, and $R(i) = \emptyset$ otherwise.

We prove Theorem 8 by showing the following claims:

**Claim 4.** *For any integer* $i$ $(0 \leqslant i \leqslant m)$, $F(i)$ *is a consistent fault set for* $\sigma_\Psi$.

**Claim 5.** $\bigcap_{i=0}^{m} F(i) = \emptyset$.

**Claim 6.** $|F(i)| = \mathrm{O}(\sqrt{kN})$ *for any integer* $i$ $(0 \leqslant i \leqslant m)$.

*5.2.1.3. Proof of Claim 4.* We prove the claim by showing that neither (i) nor (ii) below holds for any $i = 0, 1, \ldots, m$;

(i) $\sigma_\Psi\langle x, y\rangle = 0$ if $x \in V(\widehat{T}_{k,N}) - F(i)$ and $y \in F(i)$,
(ii) $\sigma_\Psi\langle x, y\rangle = 1$ if $x, y \in V(\widehat{T}_{k,N}) - F(i)$.

Let $F(i)$ be a fault set. Let $x \in V(\widehat{T}_{k,N}) - F(i)$ and $\langle x, y\rangle \in A(\widehat{T}_{k,N})$.

*Case* 1: $x \in P(j)(\neq R(i))$ for some $j \neq i$: The vertices adjacent to $x$ are contained in $P(j) \cup (\mathscr{Q} - \{q_i\})$.

*Case* 1.1: $y \in F(i)$: $y \in \mathscr{Q} - \{q_i\}$ and so $\sigma_\Psi\langle x, y\rangle = 1$.

*Case* 1.2: $y \in V(\widehat{T}_{k,N}) - F(i)$: $y \in P(j)$ and so $\sigma_\Psi\langle x, y\rangle = 0$.

*Case* 2: $x = q_i$ $(i \neq 0)$: $y \in P(i) \cup (\mathscr{Q} - \{q_i\}) \cup R(i) = F(i)$ and $\sigma_\Psi\langle x, y\rangle = 1$.

Thus, neither (i) nor (ii) holds for any arc $\langle x, y\rangle$.

*5.2.1.4. Proof of Claim 5.* The claim follows from the fact that $\mathscr{P} \cap F(0) = \emptyset$, and $q_i \notin F(i)$ for any integer $i$ $(1 \leqslant i \leqslant m)$.

*5.2.1.5. Proof of Claim 6.* Let $V_i = P(i) \cup \{q_i\}$ for any integer $i$ $(1 \leqslant i \leqslant m)$. It is easy to see the following lemma:

**Lemma 8.** $|V_i| \leqslant k\Psi$ *for any integer* $i$ $(1 \leqslant i \leqslant m)$.

**Lemma 9.** $|V_i| \geqslant \Psi$ *for any integer* $i$ $(1 \leqslant i \leqslant m-1)$.

**Proof.** If $r'' = \mathrm{child}_T(r')$ then we have

$$|V(T(r''))| \geqslant \frac{|V(T(r'))| - 1}{k}.$$

Thus, if $|V(T(r'))| \geqslant k\Psi + 1$ then $|V(T(r''))| \geqslant \Psi$. Hence $|V_i| \geqslant \Psi$ for any integer $i$ $(1 \leqslant i \leqslant m-1)$. □

By Lemma 8, we have

$$|P(i)| \leqslant k\Psi - 1$$

for any integer $i$ $(1 \leqslant i \leqslant m)$. Thus, for any integer $i$ $(0 \leqslant i \leqslant m)$,

$$|F(i)| \leqslant \begin{cases} m & \text{if } i = 0, \\ 2k\Psi + m - 3 & \text{otherwise.} \end{cases}$$

Since $m \leqslant \lceil N/\Psi \rceil$ by Lemma 9, we have

$$|F(i)| \leqslant 2k\Psi + \frac{N}{\Psi} - 2.$$

By setting $\Psi = \lceil \sqrt{N/2k} \rceil$, we conclude that, for any integer $i$ $(0 \leqslant i \leqslant m)$,

$$|F(i)| = \mathrm{O}(\sqrt{kN}),$$

which completes the proof.

### 5.3. *Complete k-ary trees of even height*

The complete $k$-ary tree of height $h$, denoted by $T_k(h)$, is defined as the $k$-ary tree of height $h$ such that every vertex $v$ with $l_{T_k(h)}(v) < h$ has exactly $k$ children.

In this subsection, we prove the following upper bound:

**Theorem 9.** *Let $k$ be an integer with $k \geqslant 2$. Then,*

$$\delta(\widehat{T_k}(h)) = \mathrm{O}(\sqrt{N}),$$

*where $N = (k^{h+1} - 1)/(k - 1)$.*

The following corollary is a direct consequence of Theorems 3 and 9.

**Corollary 4.** $\delta(\widehat{T_k}(h)) = \Theta(\sqrt{N})$ *for any positive integer $k \geqslant 2$.*

#### 5.3.1. *Proof of Theorem 9*

*5.3.1.1. Partition of $V(\widehat{T_k}(h))$.* Let $\Psi$ be a positive integer such that $\Psi \leqslant h - 1$. For any integer $i$ $(1 \leqslant i \leqslant k^{\Psi})$, define that

$$P(i) = \{x : x \text{ is a descendant of } q_i \text{ and } x \neq q_i\},$$

where $q_1, q_2, \ldots, q_{k^\Psi}$ denote $k^\Psi$ vertices of level $\Psi$ in $T_k(h)$. Let $\mathscr{P} = \bigcup_i P(i)$,

$$\mathscr{Q} = \{q_1, q_2, \ldots, q_{k^\Psi}\}, \quad \text{and} \quad \mathscr{R} = \{x : 0 \leqslant l_{T_k(h)}(x) \leqslant \Psi - 1\}.$$

It is easy to see that $(P(1), \ldots, P(k^\Psi), \mathscr{Q}, \mathscr{R})$ is a partition of $V(T_k(h))$, and thus a partition of $V(\widehat{T_k}(h))$.

*5.3.1.2. Syndrome and fault sets.* The syndrome $\sigma_\Psi$ for $\widehat{T_k}(h)$ is defined as follows:

$$\sigma_\Psi\langle x, y\rangle = \begin{cases} 1 & \text{if } x \in \mathscr{Q} \text{ or } y \in \mathscr{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

We define $k^\Psi + 1$ fault sets as follows: For any integer $i$ $(0 \leqslant i \leqslant k^\Psi)$,

$$F(i) = \begin{cases} \mathscr{Q} & \text{if } i = 0, \\ P(i) \cup (\mathscr{Q} - \{q_i\}) \cup \mathscr{R} & \text{otherwise.} \end{cases}$$

We prove Theorem 9 by showing the following claims:

**Claim 7.** *$F(i)$ is a consistent fault set for $\sigma_\Psi$ for any integer $i$ $(0 \leqslant i \leqslant k^\Psi)$.*

**Claim 8.** $\bigcap_{i=0}^{k^{\Psi}} F(i)=\emptyset$.

**Claim 9.** $|F(i)|=\mathrm{O}(\sqrt{N})$ *for any integer* $i$ $(0 \leqslant i \leqslant k^{\Psi})$.

*5.3.1.3. Proof of Claim 7.* We prove the claim by showing that neither (i) nor (ii) below holds for any $i=0,1,\ldots,k^{\Psi}$;

(i) $\sigma_{\Psi}\langle x,y\rangle=0$ if $x\in V(\widehat{T}_k(h))-F(i)$ and $y\in F(i)$,
(ii) $\sigma_{\Psi}\langle x,y\rangle=1$ if $x,y\in V(\widehat{T}_k(h))-F(i)$.

Let $F(i)$ be a fault set, $x\in V(\widehat{T}_k(h))-F(i)$, and $\langle x,y\rangle\in A(\widehat{T}_k(h))$.

*Case* 1: $x\in P(j)$ for some $j\neq i$: The vertices adjacent to $x$ are contained in $P(j)\cup(\mathscr{Q}-\{q_i\})$.

*Case* 1.1: $y\in F(i)$: $y\in\mathscr{Q}-\{q_i\}$ and so $\sigma_{\Psi}\langle x,y\rangle=1$.

*Case* 1.2: $y\in V(\widehat{T}_k(h))-F(i)$: $y\in P(j)$ and so $\sigma_{\Psi}\langle x,y\rangle=0$.

*Case* 2: $x=q_i$ $(i\neq 0)$: $y\in P(i)\cup\mathscr{R}\subset F(i)$ and $\sigma_{\Psi}\langle x,y\rangle=1$.

*Case* 3: $x\in\mathscr{R}$ $(i=0)$: The vertices adjacent to $x$ are contained in $\mathscr{Q}\cup\mathscr{R}$.

*Case* 3.1: $y\in F(i)$: $y\in\mathscr{Q}$ and so $\sigma_{\Psi}\langle x,y\rangle=1$.

*Case* 3.2: $y\in V(\widehat{T}_k(h))-F(i)$: $y\in\mathscr{R}$ and so $\sigma_{\Psi}\langle x,y\rangle=0$.

Thus, neither (i) nor (ii) holds for any arc $\langle x,y\rangle$.

*5.3.1.4. Proof of Claim 8.* The claim follows from the fact that $(\mathscr{P}\cup\mathscr{R})\cap F(0)=\emptyset$, and $q_i\notin F(i)$ for any integer $i$ $(1\leqslant i\leqslant k^{\Psi})$.

*5.3.1.5. Proof of Claim 9.* For any integer $i$ $(1\leqslant i\leqslant k^{\Psi})$

$$|P(i)|=k\cdot\frac{k^{h-\Psi}-1}{k-1},\quad |\mathscr{Q}|=k^{\Psi},\quad \text{and}\quad |\mathscr{R}|=\frac{k^{\Psi}-1}{k-1}.$$

Thus,

$$|F(i)|=\begin{cases}k^{\Psi} & \text{if } i=0,\\ k(k^{\Psi}+k^{h-\Psi}-2)/(k-1) & \text{otherwise,}\end{cases}$$

for any integer $i$ $(0\leqslant i\leqslant k^{\Psi})$. If we choose $\Psi=h/2$, we have

$$|F(i)|=\mathrm{O}(\sqrt{N}).$$

## 6. Hypercubes

The $n$-dimensional cube, denoted by $Q_n$, is defined as follows:

$$V(Q_n)=[2]^n;\quad E(Q_n)=\{(\boldsymbol{x},\boldsymbol{y}):d_{\mathrm{H}}(\boldsymbol{x},\boldsymbol{y})=1\},$$

where $d_{\mathrm{H}}(\boldsymbol{x},\boldsymbol{y})$ denotes the Hamming distance between $\boldsymbol{x}$ and $\boldsymbol{y}$. Let $N=|V(Q_n)|=2^n$. The following lower bound can be found in the literature.

**Theorem 10** (*Khanna and Fuchs [6]*). $\delta(\widehat{Q}_n) = \Omega\left(\frac{N \log\log N}{\log N}\right)$.

In this section, we prove the following lower and upper bounds:

**Theorem 11.** $\delta(\widehat{Q}_n) = \Omega\left(\frac{N}{\sqrt{\log N}}\right)$.

**Theorem 12.** $\delta(\widehat{Q}_n) = \mathrm{O}\left(\frac{N \log\log N}{\sqrt{\log N}}\right)$.

### 6.1. Proof of Theorem 11

Kleitman proves in [7] the following theorem on the $k$-partition number of the $n$-dimensional cube:

**Theorem 13.** $\Upsilon_{Q_n}(2^{n-1}+1) \geqslant \binom{n}{\lfloor n/2 \rfloor} - 1$.

The following lemma is well-known. (See [1].)

**Lemma 10** (*Cormen et al. [1]*). $\binom{n}{\lfloor n/2 \rfloor} = \Theta\left(\frac{2^n}{\sqrt{n}}\right)$.

By combining Theorems 4 and 13, and Lemma 10, we have

$$\delta(\widehat{Q}_n) = \Omega\left(\frac{N}{\sqrt{\log N}}\right),$$

where $N = 2^n$.

### 6.2. Proof of Theorem 12

#### 6.2.1. The case when n is a power of 2

*6.2.1.1. Partition of $V(\widehat{Q}_n)$.* Let $k$ be a non-negative integer. $\mathrm{bin}(k, m)$ is the $m$-bit binary representation of $k$, and $\mathrm{bin}(k, m, i)$ is the $i$th least significant bit of $\mathrm{bin}(k, m)$ $(0 \leqslant k \leqslant 2^m - 1, 1 \leqslant i \leqslant m)$. If $\boldsymbol{x} = \mathrm{bin}(k, m)$ then we denote $k = \mathrm{dec}(\boldsymbol{x})$. Let $\Psi$ be an integer such that $1 \leqslant \Psi \leqslant \log n$, and let $\Phi = 2^{\Psi}$. The concatenation of binary strings $\boldsymbol{x}$ and $\boldsymbol{y}$ is denoted by $\boldsymbol{x} \cdot \boldsymbol{y}$. The concatenation of $m$ $\boldsymbol{x}$'s is denoted by $\boldsymbol{x}^m$. For an integer $a$ such that $1 \leqslant a \leqslant \Psi$, $\boldsymbol{r}(a)$ is a binary string of length $n$ defined as follows:

$$\boldsymbol{r}(a) = (0^{n/2^{\Psi-a+1}} \cdot 1^{n/2^{\Psi-a+1}})^{2^{\Psi-a}}.$$

We consider $\boldsymbol{r}(a)$ as a vertex of $\widehat{Q}_n$ in a natural way. Define $p_a(\boldsymbol{x})$ and $q_a(\boldsymbol{x})$ as follows:

$$p_a(\boldsymbol{x}) = \begin{cases} 0 & \text{if } 0 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n/2 - 2, \\ 1 & \text{if } n/2 + 2 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n, \\ -1 & \text{if } n/2 - 1 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n/2 + 1, \end{cases}$$

$$q_a(\boldsymbol{x}) = \begin{cases} 0 & \text{if } 0 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n/2 - 1, \\ 1 & \text{if } n/2 + 1 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n, \\ -1 & \text{if } d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2. \end{cases}$$

It should be noted that if $p_a(\boldsymbol{x}) \in \{0, 1\}$ then $q_a(\boldsymbol{x}) = p_a(\boldsymbol{x})$ by definition.

For an integer $b$ such that $0 \leqslant b \leqslant \Phi - 1$, define subsets $P(b)$, $Q(b)$, and $R(b)$ of $V(\widehat{Q}_n)$ as follows:

$$P(b) = \{\boldsymbol{x} : (\forall a)[p_a(\boldsymbol{x}) \in \{0, 1\}], \mathrm{dec}(p_\Psi(\boldsymbol{x}) \cdots p_1(\boldsymbol{x})) = b\},$$

$$Q(b) = \{\boldsymbol{x} : (\exists a')[p_{a'}(\boldsymbol{x}) = -1], (\forall a)[q_a(\boldsymbol{x}) \in \{0, 1\}], \mathrm{dec}(q_\Psi(\boldsymbol{x}) \cdots q_1(\boldsymbol{x})) = b\},$$

$$R(b) = \{\boldsymbol{x} : (\exists a)[q_a(\boldsymbol{x}) = -1], T(\boldsymbol{x}) = b\},$$

where $T(\boldsymbol{x})$ is the decimal representation of the most significant $\Psi$ bits of $\boldsymbol{x}$. Define $\mathscr{P} = \bigcup_b P(b)$, $\mathscr{Q} = \bigcup_b Q(b)$, and $\mathscr{R} = \bigcup_b R(b)$.

**Lemma 11.** *$\Pi = (P(0), \ldots, P(\Phi - 1), Q(0), \ldots, Q(\Phi - 1), R(0), \ldots, R(\Phi - 1))$ is a partition of $V(\widehat{Q}_n)$.*

**Proof.** We will prove the lemma by showing the following:

(i) for any distinct blocks $U$ and $U'$ of $\Pi$, $U \cap U' = \emptyset$;
(ii) $\mathscr{P} \cup \mathscr{Q} \cup \mathscr{R} = V(\widehat{Q}_n)$.

**Proof of (i).** First of all, observe that $\mathscr{P} \cap \mathscr{Q} = \mathscr{Q} \cap \mathscr{R} = \mathscr{R} \cap \mathscr{P} = \emptyset$ by definition. We will show that $P(b) \cap P(b') = \emptyset$ for any distinct $b$ and $b'$ ($0 \leqslant b, b' \leqslant \Phi - 1$). Assume contrary that $P(b) \cap P(b') \neq \emptyset$ for some distinct $b$ and $b'$. There exists $a$ such that $\mathrm{bin}(b, \Psi, a) \neq \mathrm{bin}(b', \Psi, a)$. Suppose without loss of generality that $\mathrm{bin}(b, \Psi, a) = 0$ and $\mathrm{bin}(b', \Psi, a) = 1$. Let $\boldsymbol{x} \in P(b) \cap P(b')$. Since $\boldsymbol{x} \in P(b)$, we have $p_a(\boldsymbol{x}) = 0$ and $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n/2 - 2$. However, since $\boldsymbol{x} \in P(b')$, we also have $p_a(\boldsymbol{x}) = 1$ and $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \geqslant n/2 + 2$, a contradiction. Thus, $P(b) \cap P(b') = \emptyset$ for any distinct $b$ and $b'$. Similarly, it can be shown that $Q(b) \cap Q(b')$ for any distinct $b$ and $b'$. It is easy to see that $R(b) \cap R(b')$ for any distinct $b$ and $b'$.

**Proof of (ii).** Suppose $\boldsymbol{x} \in V(\widehat{Q}_n)$. For any $a$ such that $1 \leqslant a \leqslant \Psi$, we have $0 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n$. If $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2$ for some $a$ then $q_a(\boldsymbol{x}) = -1$, and so $\boldsymbol{x} \in R(b)$ for $b$ with $T(\boldsymbol{x}) = b$. If $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \neq n/2$ for any $a$ and $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a')) = n/2 \pm 1$ for some $a'$ then $q_a(\boldsymbol{x}) \in \{0, 1\}$ and $p_{a'}(\boldsymbol{x}) = -1$, and so $\boldsymbol{x} \in Q(b)$ for $b$ with $\mathrm{dec}(q_\Psi(\boldsymbol{x}) \cdots q_1(\boldsymbol{x})) = b$. If $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \notin \{n/2, n/2 \pm 1\}$ for any $a$ then $p_a(\boldsymbol{x}) \in \{0, 1\}$, and so $\boldsymbol{x} \in P(b)$ for $b$ with $\mathrm{dec}(p_\Psi(\boldsymbol{x}) \cdots p_1(\boldsymbol{x})) = b$. Thus, we conclude that if $\boldsymbol{x} \in V(\widehat{Q}_n)$ then $\boldsymbol{x} \in \mathscr{P} \cup \mathscr{Q} \cup \mathscr{R}$ and we have $V(\widehat{Q}_n) = \mathscr{P} \cup \mathscr{Q} \cup \mathscr{R}$. □

*6.2.1.2. Syndrome and fault sets.* The syndrome $\sigma_\Phi$ for $\widehat{Q}_n$ is defined as follows:

$$\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \begin{cases} 0 & \text{if } \begin{cases} 1. & \boldsymbol{x}, \boldsymbol{y} \in P(b) \text{ for some } b, \\ 2. & \boldsymbol{x} \in Q(b) \text{ and } \boldsymbol{y} \in R(b) \text{ for some } b, \text{ or} \\ 3. & \boldsymbol{x} \in R(b) \text{ and } \boldsymbol{y} \in Q(b) \text{ for some } b, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define $\Phi$ fault sets as follows: For any integer $b$ $(0 \leqslant b \leqslant \Phi - 1)$,

$$F(b) = P(b) \cup (\mathscr{Q} - Q(b)) \cup (\mathscr{R} - R(b)).$$

We prove Theorem 12 by showing the following claims.

**Claim 10.** *For any integer $b$ $(0 \leqslant b \leqslant \Phi - 1)$, $F(b)$ is a consistent fault set for $\sigma_\Phi$.*

**Claim 11.** $\bigcap_{b=0}^{\Phi-1} F(b) = \emptyset$.

**Claim 12.** $|F(b)| = \mathrm{O}\left(\frac{N \log \log N}{\sqrt{\log N}}\right)$ *for any integer $b$ $(0 \leqslant b \leqslant \Phi - 1)$.*

*6.2.1.3. Proof of Claim 10.* Before proving the claim, we need a couple of lemmas.

**Lemma 12.** *For any adjacent vertices $\boldsymbol{x}, \boldsymbol{y} \in V(\widehat{Q}_n)$,*

(1) *if $\boldsymbol{x} \in \mathscr{Q}$ then $\boldsymbol{y} \notin \mathscr{Q}$.*
(2) *if $\boldsymbol{x} \in \mathscr{R}$ then $\boldsymbol{y} \notin \mathscr{R}$.*

**Proof.** We will show (1). Assume contrary that $\boldsymbol{x}, \boldsymbol{y} \in \mathscr{Q}$. Then, there exists $a$ and $a'$ such that

$$p_a(\boldsymbol{x}) = -1, \quad q_a(\boldsymbol{x}) \neq -1, \quad p_{a'}(\boldsymbol{y}) = -1, \quad \text{and} \quad q_{a'}(\boldsymbol{y}) \neq -1.$$

We also have

$$d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)), \quad d_{\mathrm{H}}(\boldsymbol{y}, \boldsymbol{r}(a')) = n/2 \pm 1.$$

Since

$$d_{\mathrm{H}}(\boldsymbol{r}(a), 0^n) = d_{\mathrm{H}}(\boldsymbol{r}(a'), 0^n) = n/2,$$

we conclude that

$$\begin{aligned} &d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) + d_{\mathrm{H}}(\boldsymbol{r}(a), 0^n) + d_{\mathrm{H}}(0^n, \boldsymbol{r}(a')) + d_{\mathrm{H}}(\boldsymbol{r}(a'), \boldsymbol{y}) \\ &\quad = 2n - 2, 2n, 2n + 2, \end{aligned}$$

which is even. However, since $\boldsymbol{x}$ and $\boldsymbol{y}$ are adjacent, $d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{y}) = 1$, which is odd, a contradiction.

We can show (2) by a similar argument. □

**Lemma 13.** *For any integer $b$ $(0 \leqslant b \leqslant \Phi - 1)$,*

(1) *The vertices adjacent to $\boldsymbol{x} \in P(b)$ are contained in $P(b) \cup Q(b)$.*
(2) *The vertices adjacent to $\boldsymbol{x} \in Q(b)$ are contained in $P(b) \cup \mathscr{R}$.*
(3) *The vertices adjacent to $\boldsymbol{x} \in R(b)$ are contained in $\mathscr{Q}$.*

**Proof.** We will show (1). Let $\boldsymbol{x} \in P(b)$ and $\boldsymbol{y}$ be a vertex adjacent to $\boldsymbol{x}$. Then

$$|d_{\mathrm{H}}(\boldsymbol{y}, \boldsymbol{r}(a)) - d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a))| = 1$$

for any $a$. If $p_a(\boldsymbol{x}) = 0$ then

$$0 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant \frac{n}{2} - 2.$$

Thus, we have

$$0 \leqslant d_{\mathrm{H}}(\boldsymbol{y}, \boldsymbol{r}(a)) \leqslant \frac{n}{2} - 1,$$

and so $q_a(\boldsymbol{y}) = 0$. If $p_a(\boldsymbol{x}) = 1$ then

$$\frac{n}{2} + 2 \leqslant d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) \leqslant n.$$

It follows that

$$\frac{n}{2} + 1 \leqslant d_{\mathrm{H}}(\boldsymbol{y}, \boldsymbol{r}(a)) \leqslant n,$$

and so $q_a(\boldsymbol{y})=1$. Thus we conclude that $q_a(\boldsymbol{y})=p_a(\boldsymbol{x})$ for any $a$ and so $\mathrm{dec}(q_\Psi(\boldsymbol{y}) \cdots q_1(\boldsymbol{y}))=b$. If there exists $a'$ such that $d_{\mathrm{H}}(\boldsymbol{y}, \boldsymbol{r}(a'))=n/2 \pm 1$ then $\boldsymbol{y} \in Q(b)$. Otherwise, $p_a(\boldsymbol{y})=q_a(\boldsymbol{y})$ for any $a$, and so $\boldsymbol{y} \in P(b)$.

(2) and (3) follow from (1) and Lemma 12. □

We will prove Claim 10 by showing that neither (i) nor (ii) below holds for any $b$:

(i) $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$ if $\boldsymbol{x} \in V(\widehat{Q}_n) - F(b)$ and $\boldsymbol{y} \in F(b)$,
(ii) $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$ if $\boldsymbol{x}, \boldsymbol{y} \in V(\widehat{Q}_n) - F(b)$.

Let $F(b)$ be a fault set. Let $\boldsymbol{x} \in V(\widehat{Q}_n) - F(b)$ and $\langle \boldsymbol{x}, \boldsymbol{y}\rangle \in A(\widehat{Q}_n)$.

*Case* 1: $\boldsymbol{x} \in P(b')$ for some $b' \neq b$: From Lemma 13, the vertices adjacent to $\boldsymbol{x}$ are contained in $P(b') \cup Q(b')$.

*Case* 1.1: $\boldsymbol{y} \in F(b)$: $\boldsymbol{y} \in Q(b')$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$.

*Case* 1.2: $\boldsymbol{y} \in V(\widehat{Q}_n) - F(b)$: $\boldsymbol{y} \in P(b')$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

*Case* 2: $\boldsymbol{x} \in Q(b)$: From Lemma 13, the vertices adjacent to $\boldsymbol{x}$ are contained in $P(b) \cup \mathscr{R}$.

*Case* 2.1: $\boldsymbol{y} \in F(b)$: $\boldsymbol{y} \notin R(b)$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$.

*Case* 2.2: $\boldsymbol{y} \in V(\widehat{Q}_n) - F(b)$: $\boldsymbol{y} \in R(b)$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

*Case* 3: $\boldsymbol{x} \in R(b)$: From Lemma 13, the vertices adjacent to $\boldsymbol{x}$ are contained in $\mathscr{Q}$.

*Case* 3.1: $\boldsymbol{y} \in F(b)$: $\boldsymbol{y} \notin Q(b)$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 1$.

*Case* 3.2: $\boldsymbol{y} \in V(\widehat{Q}_n) - F(b)$: $\boldsymbol{y} \in Q(b)$ and so $\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.

Thus, neither (i) nor (ii) holds for any arc $\langle \boldsymbol{x}, \boldsymbol{y}\rangle$.

*6.2.1.4. Proof of Claim 11.* The claim follows from the fact that $Q(b) \cap F(b) = R(b) \cap F(b) = \emptyset$ for any $b$, and $P(b) \cap F(b') = \emptyset$ for any distinct $b$ and $b'$.

*6.2.1.5. Proof of Claim 12.* We will prove the claim by a series of lemmas.

**Lemma 14.** $|\mathscr{Q}| < 2\Psi \cdot \binom{n}{n/2-1}$.

**Proof.**

$$
\begin{aligned}
|\mathcal{Q}| &< |\{\boldsymbol{x} : (\exists a)[d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2 \pm 1]\}| \\
&< |\{\boldsymbol{x} : (\exists a)[d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2 - 1]\}| \\
&\quad + |\{\boldsymbol{x} : (\exists a)[d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2 + 1]\}| \\
&< \sum_{i=1}^{\Psi} |\{\boldsymbol{x} : d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(i)) = n/2 - 1\}| + \sum_{i=1}^{\Psi} |\{\boldsymbol{x} : d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(i)) = n/2 + 1\}| \\
&= 2 \sum_{i=1}^{\Psi} \binom{n}{n/2-1} = 2\Psi \cdot \binom{n}{n/2-1}. \qquad \square
\end{aligned}
$$

**Lemma 15.** $|\mathcal{R}| < \Psi \cdot \binom{n}{n/2}$.

**Proof.**

$$
\begin{aligned}
|\mathcal{R}| &= |\{\boldsymbol{x} : (\exists a)[d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = n/2]\}| < \sum_{i=1}^{\Psi} |\{\boldsymbol{x} : d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(i)) = n/2\}| \\
&= \sum_{i=1}^{\Psi} \binom{n}{n/2} = \Psi \cdot \binom{n}{n/2}. \qquad \square
\end{aligned}
$$

**Lemma 16.** $|P(b)| = |P(b')|$ *for any integers* $b$ *and* $b'$ $(0 \leqslant b, b' \leqslant \Phi - 1)$.

**Proof.** For any integer $k$ and $a$ $(0 \leqslant k \leqslant 2^{\Psi} - 1,\ 1 \leqslant a \leqslant \Psi)$, let $\mathrm{ex}(k, a)$ denote the integer such that $\mathrm{bin}(\mathrm{ex}(k, a), \Psi)$ and $\mathrm{bin}(k, \Psi)$ differ just in the $a$th least significant bit. It should be noted that $b = \mathrm{ex}(\mathrm{ex}(b, a), a)$.

We prove the lemma by showing the following:

**Claim A.** $|P(b)| = |P(\mathrm{ex}(b, a))|$ *for any integers* $b$ *and* $a$ $(0 \leqslant b \leqslant \Phi - 1,\ 1 \leqslant a \leqslant \Psi)$.

**Proof of Claim A.** Before proving the claim, we need some preliminaries. For any $\boldsymbol{x} \in V(\widehat{Q}_n)$ and any integer $u$ $(0 \leqslant u \leqslant 2^{\Psi} - 1)$, let

$$\boldsymbol{x}_u = (x_{n/2^{\Psi} \times (u+1)}, \ldots, x_{n/2^{\Psi} \times u+1}).$$

For any distinct $a$ and $a'$ $(1 \leqslant a, a' \leqslant \Psi)$ and $w, w' \in \{0, 1\}$, let

$$W_{awa'w'}(\boldsymbol{x}) = \sum \{w_{\mathrm{H}}(\boldsymbol{x}_u) : \mathrm{bin}(u, \Psi, a) = w \text{ and } \mathrm{bin}(u, \Psi, a') = w'\},$$

$$
\begin{aligned}
W_{aw}(\boldsymbol{x}) &= \sum \{w_{\mathrm{H}}(\boldsymbol{x}_u) : \mathrm{bin}(u, \Psi, a) = w\} \\
&= W_{awa'1}(\boldsymbol{x}) + W_{awa'0}(\boldsymbol{x}),
\end{aligned}
$$

where $w_{\mathrm{H}}(\boldsymbol{x}_u)$ denotes the Hamming weight of $\boldsymbol{x}_u$. For any $\boldsymbol{x}$ and $a$ $(1 \leqslant a \leqslant \Psi)$, let

$$e_a(\boldsymbol{x}) = \boldsymbol{x}_{\mathrm{ex}(2^{\Psi}-1,a)} \cdot \boldsymbol{x}_{\mathrm{ex}(2^{\Psi}-2,a)} \cdots \boldsymbol{x}_{\mathrm{ex}(0,a)}.$$

It should be noted that $e_a$ is a one-to-one mapping and that

$$W_{a1a'w}(e_a(\boldsymbol{x})) = W_{a0a'w}(\boldsymbol{x}), \quad W_{a0a'w}(e_a(\boldsymbol{x})) = W_{a1a'w}(\boldsymbol{x}),$$

$$W_{a0}(e_a(\boldsymbol{x})) = W_{a1}(\boldsymbol{x}), \quad \text{and} \quad W_{a1}(e_a(\boldsymbol{x})) = W_{a0}(\boldsymbol{x})$$

for any distinct $a$ and $a'$ $(1 \leqslant a, a' \leqslant \Psi)$ and any $w \in \{0, 1\}$.

By the definition of $\boldsymbol{r}(a)$, it is easy to see the following claim:

**Claim B.** *For any a and u* $(1 \leqslant a \leqslant \Psi,\ 0 \leqslant u \leqslant 2^{\Psi} - 1)$,

$$\boldsymbol{r}(a)_u = \begin{cases} 1^{n/2^{\Psi}} & if\ \operatorname{bin}(u, \Psi, a) = 0, \\ 0^{n/2^{\Psi}} & if\ \operatorname{bin}(u, \Psi, a) = 1. \end{cases}$$

**Claim C.** *For any* $\boldsymbol{x}$ *and a* $(1 \leqslant a \leqslant \Psi)$,

$$d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) = W_{a1}(\boldsymbol{x}) + (n/2 - W_{a0}(\boldsymbol{x})).$$

**Proof of Claim C.** By the definition of $W_{aw}(\boldsymbol{x})$ and Claim B, we have

$$\begin{aligned} d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) &= \sum\{d_{\mathrm{H}}(\boldsymbol{x}_u, \boldsymbol{r}(a)_u) : \operatorname{bin}(u, \Psi, a) = 1\} \\ &\quad + \sum\{d_{\mathrm{H}}(\boldsymbol{x}_u, \boldsymbol{r}(a)_u) : \operatorname{bin}(u, \Psi, a) = 0\} \\ &= \sum\{d_{\mathrm{H}}(\boldsymbol{x}_u, 0^{n/2^{\Psi}}) : \operatorname{bin}(u, \Psi, a) = 1\} \\ &\quad + \sum\{d_{\mathrm{H}}(\boldsymbol{x}_u, 1^{n/2^{\Psi}}) : \operatorname{bin}(u, \Psi, a) = 0\} \\ &= \sum\{w_{\mathrm{H}}(\boldsymbol{x}_u) : \operatorname{bin}(u, \Psi, a) = 1\} \\ &\quad + \left(n/2 - \sum\{w_{\mathrm{H}}(\boldsymbol{x}_u) : \operatorname{bin}(u, \Psi, a) = 0\}\right) \\ &= W_{a1}(\boldsymbol{x}) + (n/2 - W_{a0}(\boldsymbol{x})). \qquad \square \end{aligned}$$

**Claim D.** *For any a and* $a'$ $(1 \leqslant a, a' \leqslant \Psi)$,

$$d_{\mathrm{H}}(e_a(\boldsymbol{x}), \boldsymbol{r}(a')) = \begin{cases} n - d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)) & if\ a' = a, \\ d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a')) & otherwise. \end{cases}$$

**Proof of Claim D.** Suppose that $a' = a$. Since $W_{a1}(e_a(\boldsymbol{x})) = W_{a0}(\boldsymbol{x})$ and $W_{a0}(e_a(\boldsymbol{x})) = W_{a1}(\boldsymbol{x})$ as mentioned earlier, we have from Claim C that

$$\begin{aligned} d_{\mathrm{H}}(e_a(\boldsymbol{x}), \boldsymbol{r}(a)) &= W_{a1}(e_a(\boldsymbol{x})) + (n/2 - W_{a0}(e_a(\boldsymbol{x}))) \\ &= W_{a0}(\boldsymbol{x}) + (n/2 - W_{a1}(\boldsymbol{x})) \\ &= n - \{W_{a1}(\boldsymbol{x}) + (n/2 - W_{a0}(\boldsymbol{x}))\} \\ &= n - d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a)). \end{aligned}$$

If $a' \neq a$ then we have from Claim C that

$$\begin{aligned} d_{\mathrm{H}}(e_a(\boldsymbol{x}), \boldsymbol{r}(a')) &= W_{a'1}(e_a(\boldsymbol{x})) + (n/2 - W_{a'0}(e_a(\boldsymbol{x}))) \\ &= W_{a'1a1}(e_a(\boldsymbol{x})) + W_{a'1a0}(e_a(\boldsymbol{x})) \\ &\quad + (n/2 - W_{a'0a1}(e_a(\boldsymbol{x})) + W_{a'0a0}(e_a(\boldsymbol{x}))) \\ &= W_{a'1a0}(\boldsymbol{x}) + W_{a'1a1}(\boldsymbol{x}) \\ &\quad + (n/2 - W_{a'0a0}(\boldsymbol{x}) + W_{a'0a1}(\boldsymbol{x})) \\ &= W_{a'1}(\boldsymbol{x}) + (n/2 - W_{a'0}(\boldsymbol{x})) \\ &= d_{\mathrm{H}}(\boldsymbol{x}, \boldsymbol{r}(a')). \qquad \square \end{aligned}$$

**Claim E.** *For any b and a* $(0 \leqslant b \leqslant \Phi - 1,\ 1 \leqslant a \leqslant \Psi)$,

$$\boldsymbol{x} \in P(b) \Rightarrow e_a(\boldsymbol{x}) \in P(\mathrm{ex}(b, a)).$$

**Proof of Claim E.** It follows from Claim D that if $\boldsymbol{x} \in P(b)$ then $p_{a'}(e_a(\boldsymbol{x})) \notin \{n/2, n/2 \pm 1\}$ for any $a'$ $(1 \leqslant a' \leqslant \Psi)$ and so $e_a(\boldsymbol{x}) \in \mathscr{P}$. It also follows from Claim D that if $\boldsymbol{x} \in P(b)$ then $p_a(e_a(\boldsymbol{x})) \neq p_a(\boldsymbol{x})$ and $p_{a'}(e_a(\boldsymbol{x})) = p_{a'}(\boldsymbol{x})$ for any distinct $a$ and $a'$ $(1 \leqslant a, a' \leqslant \Psi)$. Thus,

$$\mathrm{dec}(p_\Psi(e_a(\boldsymbol{x})) \cdots p_1(e_a(\boldsymbol{x}))) = \mathrm{dec}(p_\Psi(\boldsymbol{x}) \cdots \overline{p_a(\boldsymbol{x})} \cdots p_1(\boldsymbol{x})) = \mathrm{ex}(b, a),$$

where $\overline{v}$ is the complement of $v$. It follows if $\boldsymbol{x} \in P(b)$ then $e_a(\boldsymbol{x}) \in P(\mathrm{ex}(b, a))$ for any $b$ and $a$ $(0 \leqslant b \leqslant \Phi - 1, 1 \leqslant a \leqslant \Psi)$. $\square$

Now we are ready to prove Claim A. Since $e_a$ is a one-to-one mapping and $\mathrm{ex}(\mathrm{ex}(b, a), a) = b$ as mentioned above, we conclude from Claim E that $|P(b)| = |P(\mathrm{ex}(b, a))|$ for any $b$ and $a$ $(0 \leqslant b \leqslant \Phi - 1, 1 \leqslant a \leqslant \Psi)$. $\square$

This completes the proof of Lemma 16. $\square$

**Lemma 17.** $|P(b)| < 2^n/\Phi$ *for any b* $(0 \leqslant b \leqslant \Phi - 1)$.

**Proof.** From Lemma 16, we have

$$|P(b)| = \frac{|\mathscr{P}|}{\Phi} \tag{1}$$

for any $b$ $(0 \leqslant b \leqslant \Phi - 1)$. We also have

$$|\mathscr{P}| < |V(\widehat{Q}_n)| = 2^n, \tag{2}$$

from Lemma 11. From (1) and (2), we have the lemma. $\square$

**Lemma 18.** $|F(b)| = \mathrm{O}\left(\frac{N \log \log N}{\sqrt{\log N}}\right)$.

**Proof.** From Lemmas 14, 15, and 17,

$$\begin{aligned}|F(b)| &= |P(b)| + |\mathscr{Q} - Q(b)| + |\mathscr{R} - R(b)| < |P(b)| + |\mathscr{Q}| + |\mathscr{R}| \\ &< \frac{2^n}{\Phi} + 2\Psi\binom{n}{n/2-1} + \Psi\binom{n}{n/2}.\end{aligned}$$

From Lemma 10, we have

$$\binom{n}{n/2,}, \binom{n}{n/2-1} = \mathrm{O}\left(\frac{2^n}{\sqrt{n}}\right),$$

and thus

$$|F(b)| = \mathrm{O}\left(\frac{2^n}{\Phi} + \Psi\frac{2^n}{\sqrt{n}}\right).$$

If we choose $\Psi = \lceil (\log n)/2 - \log\log n \rceil$ then $\Phi = \Theta(\sqrt{n}/\log n)$ and we have

$$|F(b)| = \mathrm{O}\left(\frac{2^n \log n}{\sqrt{n}}\right) = \mathrm{O}\left(\frac{N \log\log N}{\sqrt{\log N}}\right). \qquad \square$$

*6.2.2. The case when n is not a power of 2*

Let $n_1 = 2^{\lfloor \log n \rfloor}$ and $n_2 = n - n_1$. Let $\Psi$ be an integer such that $1 \leqslant \Psi \leqslant \log n_1 = \lfloor \log n \rfloor$, and let $\Phi = 2^{\Psi}$. Define $\boldsymbol{r}(a)$, $p_a(\boldsymbol{x})$, and $q_a(\boldsymbol{x})$ on $\widehat{Q}_{n_1}$ as in Section 6.2.1.1. Define subsets $P(b)$, $Q(b)$, and $R(b)$ of $V(\widehat{Q}_n)$ as follows:

$$P(b) = \{\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 : (\forall a)[p_a(\boldsymbol{x}_1) \in \{0,1\}], \mathrm{dec}(p_\Psi(\boldsymbol{x}_1)\cdots p_1(\boldsymbol{x}_1)) = b\},$$

$$\begin{aligned}Q(b) = \{\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 &: (\exists a')[p_{a'}(\boldsymbol{x}_1) = -1], \\ &(\forall a)[q_a(\boldsymbol{x}_1) \in \{0,1\}], \mathrm{dec}(q_\Psi(\boldsymbol{x}_1)\cdots q_1(\boldsymbol{x}_1)) = b\},\end{aligned}$$

$$R(b) = \{\boldsymbol{x}_1 \cdot \boldsymbol{x}_2 : (\exists a)[q_a(\boldsymbol{x}_1) = -1], T(\boldsymbol{x}_1) = b\},$$

where $\boldsymbol{x}_i \in [2]^{n_i}$ ($i = 1$ or 2). Define the syndrome $\sigma_\Phi$ for $\widehat{Q}_n$ and $\Phi$ fault sets $F(b)$ as follows:

$$\sigma_\Phi\langle \boldsymbol{x}, \boldsymbol{y}\rangle = \begin{cases} 0 & \text{if } \begin{cases} 1. & \boldsymbol{x}, \boldsymbol{y} \in P(b) \text{ for some } b, \text{ or} \\ 2. & \boldsymbol{x}, \boldsymbol{y} \in Q(b) \cup R(b) \text{ for some } b, \end{cases} \\ 1 & \text{otherwise,} \end{cases}$$

$$F(b) = P(b) \cup (\mathscr{Q} - Q(b)) \cup (\mathscr{R} - R(b)) \ (0 \leqslant b \leqslant \Phi).$$

Then, we can prove that

$$\delta(\widehat{Q}_n) = \mathrm{O}\left(\frac{2^{n_1} \log n_1}{\sqrt{n_1}} 2^{n_2}\right) = \mathrm{O}\left(\frac{N \log\log N}{\sqrt{\log N}}\right)$$

by a similar arguments as those of Section 6.2.1.

## 7. Cube-connected cycles

For any $\boldsymbol{x} = x_{n-1}x_{n-2}\cdots x_0 \in [2]^n$, define

$$\chi_i(\boldsymbol{x}) = x_{n-1}\cdots x_{i+1}\overline{x_i}x_{i-1}\cdots x_0.$$

The $n$-dimensional cube-connected cycles, denoted by $CCC_n$, is the graph defined as follows:

$$V(CCC_n) = \{[\boldsymbol{x}, i] : \boldsymbol{x} \in [2]^n, i \in [n]\};$$

$$E(CCC_n) = \{([\boldsymbol{x}, i], [\boldsymbol{x}, j]) : \boldsymbol{x} \in [2]^n,\ \ j = (i \pm 1) \bmod n\} \\ \cup \{([\boldsymbol{x}, i], [\boldsymbol{y}, i]) : \boldsymbol{x} \in [2]^n,\ \boldsymbol{y} = \chi_i(\boldsymbol{x})\}.$$

$CCC_n$ is constructed from $Q_n$ by replacing each vertex of $Q_n$ with a cycle of length $n$ in $CCC_n$. It should be noted that $|V(Q_n)| = 2^n$ and $N = |V(CCC_n)| = n2^n$.

In this section, we prove the following bounds:

**Theorem 14.** $\delta(\widehat{CCC}_n) \geqslant \left\lfloor \frac{2N}{4n+1} \right\rfloor$.

**Theorem 15.** $\delta(\widehat{CCC}_n) < \frac{4N}{n} + \mathrm{o}\left(\frac{N}{n}\right)$.

From Theorems 14 and 15, we have the following corollary.

**Corollary 5.** $\delta(\widehat{CCC}_n) = \Theta\left(\frac{N}{\log N}\right)$.

### 7.1. Proof of Theorem 14

Before proving the theorem, we need some preliminaries, which are also used in Section 8.

Let $G$ be an $N$-vertex connected graph. A walk $W$ in $G$ is defined as a sequence $[v_0, v_1, \ldots, v_k]$ of vertices such that $(v_i, v_{i+1}) \in E(G)$ for any $i \in [k]$. $W$ is also called a $(v_0, v_k)$-walk. The length of $W$, denoted by $|W|$, is defined as $k$. For any ordered pair $[u, v]$ of vertices in $G$, let $W[u, v]$ be a $(u, v)$-walk in $G$. We define

$$\mathscr{W}(w) = \{W[u, v] :\ w \in V(W[u, v])\},$$

for any $w \in V(G)$, and

$$\mathscr{W}(S) = \bigcup_{w \in S} \mathscr{W}(w)$$

for any $S \subseteq V(G)$.

**Lemma 19.** *Let $t$ be a positive integer, and $F \subseteq V(G)$ with $|F| = t$. If every connected component of the subgraph $H$ of $G$ induced by the vertices in $V(G) - F$ has size $t$ or smaller then $|\mathscr{W}(F)| > N^2 - tN$.*

**Proof.** We prove the lemma by a series of claims. Let $V_1, \ldots, V_k$ be vertex sets of the connected components of $H$.

**Claim 13.** *If $F \cap V(W[u, v]) = \emptyset$ then $u, v \in V_i$ for some $i$.*

**Proof of Claim 13.** The lemma follows from the fact that the vertices adjacent to $w \in V_i$ are contained in $V_i \cup F$ for any $i$. □

**Claim 14.** *Let m be a positive integer. If $a_1, a_2, \ldots,$ and $a_k$ are integers such that* (i) $0 \leqslant a_i \leqslant t$ *for any $i$, and* (ii) $\sum_{i=1}^{k} a_i = m$, *then*

$$\sum_{i=1}^{k} a_i^2 \leqslant \left\lfloor \frac{m}{t} \right\rfloor t^2 + (m \bmod t)^2.$$

**Proof of Claim 14.** Let $b_1, \ldots, b_k$ be $k$ integers satisfying the conditions (i) and (ii) such that $\sum_{i=1}^{k} b_i^2 \geqslant \sum_{i=1}^{k} a_i^2$. Assume that $0 < b_p \leqslant b_q < t$ for some distinct positive integers $p$ and $q$. For any $i$, let

$$c_i = \begin{cases} b_i - 1 & \text{if } i = p, \\ b_i + 1 & \text{if } i = q, \\ b_i & \text{otherwise.} \end{cases}$$

Then, $c_1, \ldots, c_k$ satisfy the conditions (i) and (ii), and

$$\sum_{i=1}^{k} c_i^2 - \sum_{i=1}^{k} b_i^2 = (b_p - 1)^2 + (b_q + 1)^2 - b_p^2 - b_q^2 = 2(b_q - b_p) + 2 > 0,$$

which is a contradiction. Hence, we have at most one integer $b_p$ with $1 \leqslant b_p < t$. Assume without loss of generality that $t \geqslant b_1 \geqslant \cdots \geqslant b_k \geqslant 0$. Then,

$$b_i = \begin{cases} t & \text{if } 1 \leqslant i \leqslant \lfloor m/t \rfloor, \\ m \bmod t & \text{if } i = \lfloor m/t \rfloor + 1, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that

$$\sum_{i=1}^{k} a_i^2 \leqslant \sum_{i=1}^{k} b_i^2 = \left\lfloor \frac{m}{t} \right\rfloor t^2 + (m \bmod t)^2. \qquad \square$$

By Claim 13,

$$|\mathscr{W}(F)| \geqslant N^2 - \sum_{i=1}^{k} |V_i|^2.$$

Since $|V_i| \leqslant t$ for any $i$ and $\sum_{i=1}^{k} |V_i| = N - t$, we obtain by Claim 14 that

$$|\mathscr{W}(F)| \geqslant N^2 - \left\lfloor \frac{N-t}{t} \right\rfloor t^2 - \{(N-t) \bmod t\}^2 > N^2 - \left\lfloor \frac{N}{t} \right\rfloor t^2 \geqslant N^2 - tN.$$

This completes the proof of Lemma 19. □

Now, we are ready to prove Theorem 14.

**Lemma 20.** *For* $t = \lfloor 2N/(4n+1) \rfloor$,

$$\Upsilon_{CCC_n}(t+1) \geqslant t.$$

**Proof.** Assume contrary that $\Upsilon_{CCC_n}(t+1) < t$. Then, there exists some $F \subseteq V(CCC_n)$ with $|F| = t$ such that every connected component of the subgraph $G$ of $CCC_n$ induced by the vertices in $V(CCC_n) - F$ has size $t$ or smaller.

For any ordered pair of vertices $u = [x_{n-1}x_{n-2}\cdots x_0, i]$ and $v = [y_{n-1}y_{n-2}\cdots y_0, j]$ in $V(CCC_n)$, let $W[u, v]$ denote the following walk connecting $u$ and $v$ in $CCC_n$:

$$\begin{array}{lllll}
u & = & [x_{n-1}\cdots x_{i+1}x_i x_{i-1}\cdots x_0, i] & \to & [x_{n-1}\cdots x_{i+1}y_i x_{i-1}\cdots x_0, i] \\
 & \to & [x_{n-1}\cdots x_{i+1}y_i x_{i-1}\cdots x_0, i+1] & \to & [x_{n-1}\cdots x_{i+2}y_{i+1}y_i x_{i-1}\cdots x_0, i+1] \\
 & \to & \cdots & \to & [y_{n-1}y_{n-2}\cdots y_i x_{i-1}\cdots x_0, n-1] \\
 & \to & [y_{n-1}y_{n-2}\cdots y_i x_{i-1}\cdots x_0, 0] & \to & [y_{n-1}y_{n-2}\cdots y_i x_{i-1}\cdots y_0, 0] \\
 & \to & \cdots & \to & [y_{n-1}y_{n-2}\cdots y_i y_{i-1}\cdots y_0, i-1] \\
 & \to & [y_{n-1}y_{n-2}\cdots y_i y_{i-1}\cdots y_0, i] & \to & \cdots \\
 & \to & [y_{n-1}y_{n-2}\cdots y_i y_{i-1}\cdots y_0, j] & = & v.
\end{array}$$

For any $w \in V(CCC_n)$, define $\mathscr{W}(w) = \{W[u, v] : u, v \in V(CCC_n), w \in V(W[u, v])\}$, and for any $V \subseteq V(CCC_n)$, define $\mathscr{W}(V) = \bigcup_{w \in V} \mathscr{W}(w)$.

**Lemma 21.** *For any* $w \in V(CCC_n)$,

$$|\mathscr{W}(w)| \leqslant \left(2n - \tfrac{1}{2}\right) N.$$

*Moreover,*

$$|\mathscr{W}(F)| \leqslant t \left(2n - \tfrac{1}{2}\right) N.$$

**Proof.** It is easy to see that

$$\sum_{w \in V(CCC_n)} |\mathscr{W}(w)| \leqslant \sum_{u,v \in V(CCC_n)} (|W[u, v]| + 1), \quad \text{and}$$

$$\sum_{u,v \in V(CCC_n)} |V(W[u, v])| = \left(n - 1 + \frac{n}{2} + \frac{n-1}{2} + 1\right) N^2 = \left(2n - \frac{1}{2}\right) N^2.$$

Thus, by the symmetry of $\mathscr{W}(w)$, we have

$$|\mathscr{W}(w)| \leqslant \left(2n - \tfrac{1}{2}\right) N,$$

and

$$|\mathscr{W}(F)| \leqslant \sum_{w \in F} |\mathscr{F}(w)| \leqslant t\left(2n - \tfrac{1}{2}\right)N. \qquad \square$$

By Lemma 19, we have

$$|\mathscr{W}(F)| > N^2 - \frac{2N}{4n+1}N = \frac{4n-1}{4n+1}N^2.$$

On the other hand, by Lemma 21,

$$|\mathscr{W}(F)| \leqslant \frac{2N}{4n+1}\left(2n - \frac{1}{2}\right)N = \frac{4n-1}{4n+1}N^2,$$

which is a contradiction. Hence,

$$\Upsilon_{CCC_n}(t+1) \geqslant t. \qquad \square$$

From Theorem 4 and Lemma 20, we have Theorem 14.

### 7.2. *Proof of Theorem 15*

#### 7.2.1. *Partition of $V(\widehat{CCC}_n)$*

Suppose that $n$ is even for simplicity of argument. We can prove the theorem by a similar argument for odd $n$. For any $\boldsymbol{x} = x_{n-1}x_{n-2}\cdots x_0 \in [2]^n$ and positive integer $m \leqslant n$, define mappings $L_m$ and $R_m$ as follows:

$$L_m(\boldsymbol{x}) = x_{n-1}\cdots x_{n-m} \quad \text{and} \quad R_m(\boldsymbol{x}) = x_{m-1}\ldots x_0.$$

For each $\boldsymbol{t} \in [2]^{n/2}$ and $k \in [2]$, define $P(\boldsymbol{t}, k)$ and $Q(\boldsymbol{t}, k)$ as follows:

$$P(\boldsymbol{t}, 0) = \{[\boldsymbol{x}, i] \in V(\widehat{CCC}_n) : L_{n/2}(\boldsymbol{x}) = \boldsymbol{t},\, 1 \leqslant i \leqslant (n/2) - 2\},$$

$$P(\boldsymbol{t}, 1) = \{[\boldsymbol{x}, i] \in V(\widehat{CCC}_n) : R_{n/2}(\boldsymbol{x}) = \boldsymbol{t},\, (n/2) + 1 \leqslant i \leqslant n - 2\},$$

$$Q(\boldsymbol{t}, 0) = \{[\boldsymbol{x}, i] \in V(\widehat{CCC}_n) : L_{n/2}(\boldsymbol{x}) = \boldsymbol{t},\, i = 0 \text{ or } (n/2) - 1\},$$

$$Q(\boldsymbol{t}, 1) = \{[\boldsymbol{x}, i] \in V(\widehat{CCC}_n) : R_{n/2}(\boldsymbol{x}) = \boldsymbol{t},\, i = n/2 \text{ or } n - 1\}.$$

It is easy to see that $(P(0\cdots 0, 0), \ldots, P(1\cdots 1, 1), Q(0\cdots 0, 0), \ldots, Q(1\cdots 1, 1))$ is a partition of $V(\widehat{CCC}_n)$. Let $\mathscr{P} = \bigcup_{\boldsymbol{t},k} P(\boldsymbol{t}, k)$ and $\mathscr{Q} = \bigcup_{\boldsymbol{t},k} Q(\boldsymbol{t}, k)$.

#### 7.2.2. *Syndrome and fault sets*

The syndrome $\sigma$ for $\widehat{CCC}_n$ is defined as follows:

$$\sigma\langle u, v\rangle = \begin{cases} 0 & \text{if } \begin{cases} 1. & u, v \in P(\boldsymbol{t}, k) \text{ for some } \boldsymbol{t} \text{ and } k, \text{ or} \\ 2. & u, v \in Q(\boldsymbol{t}, k) \text{ for some } \boldsymbol{t} \text{ and } k, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define $2^{(n/2)+1}$ fault sets as follows:

$$F(\boldsymbol{t},k)=P(\boldsymbol{t},k)\cup(\mathscr{Q}-Q(\boldsymbol{t},k))\quad(\boldsymbol{t}\in[2]^{n/2},\ k\in[2])$$

We prove Theorem 15 by showing the following claims:

**Claim 15.** *For any $\boldsymbol{t}\in[2]^{n/2}$ and $k\in[2]$, $F(\boldsymbol{t},k)$ is a consistent fault set for $\sigma$.*

**Claim 16.** $\bigcap_{\boldsymbol{t}\in[2]^{n/2},k\in[2]}F(\boldsymbol{t},k)=\emptyset$.

**Claim 17.** $|F(\boldsymbol{t},k)|<\frac{4N}{n}+\mathrm{o}\left(\frac{N}{n}\right)$ *for any $\boldsymbol{t}\in[2]^{n/2}$ and $k\in[2]$.*

*7.2.3. Proof of Claim 15*

Before proving the lemma, we need the following lemma.

**Lemma 22.** *For any $\boldsymbol{t}\in[2]^{n/2}$ and $k\in[2]$,*

(1) *The vertices adjacent to $u\in P(\boldsymbol{t},k)$ are contained in $P(\boldsymbol{t},k)\cup Q(\boldsymbol{t},k)$.*
(2) *The vertices adjacent to $u\in Q(\boldsymbol{t},k)$ are contained in $P(\boldsymbol{t},k)\cup\mathscr{Q}$.*

**Proof.** We will prove (1). Let $u=[\boldsymbol{x},i]\in P(\boldsymbol{t},k)$, and $v=[\boldsymbol{y},j]$ be a vertex adjacent to $u$. If $k=0$ then $1\leqslant i\leqslant(n/2)-2$, and so $L_{n/2}(\chi_i(\boldsymbol{x}))=L_{n/2}(\boldsymbol{x})$. Thus, we conclude that $L_{n/2}(\boldsymbol{y})=L_{n/2}(\boldsymbol{x})$ and $0\leqslant j\leqslant(n/2)-1$, and hence $v\in P(\boldsymbol{t},0)\cup Q(\boldsymbol{t},0)$. If $k=1$ then $(n/2)+1\leqslant i\leqslant n-2$, and so $R_{n/2}(\chi_i(\boldsymbol{x}))=R_{n/2}(\boldsymbol{x})$. Thus, we conclude that $R_{n/2}(\boldsymbol{y})=R_{n/2}(\boldsymbol{x})$ and $n/2\leqslant j\leqslant n-1$, and hence $v\in P(\boldsymbol{t},1)\cup Q(\boldsymbol{t},1)$.
2 follows from 1. $\square$

We prove Claim 15 by showing that neither (i) nor (ii) below holds for any $\boldsymbol{t}\in[2]^{n/2}$ and $k\in[2]$;

(i) $\sigma\langle u,v\rangle=0$ if $u\in V(\widehat{CCC}_n)-F(\boldsymbol{t},k)$ and $v\in F(\boldsymbol{t},k)$,
(ii) $\sigma\langle u,v\rangle=1$ if $u,v\in V(\widehat{CCC}_n)-F(\boldsymbol{t},k)$.

Let $F(\boldsymbol{t},k)$ be a fault set. Let $u\in V(\widehat{CCC}_n)-F(\boldsymbol{t},k)$ and $\langle u,v\rangle\in A(\widehat{CCC}_n)$.

*Case* 1: $u\in P(\boldsymbol{t}',k')$ for some $(\boldsymbol{t}',k')\neq(\boldsymbol{t},k)$: From Lemma 22, the vertices adjacent to $u$ are contained in $P(\boldsymbol{t}',k')\cup Q(\boldsymbol{t}',k')$.

*Case* 1.1: $v\in F(\boldsymbol{t},k)$: $v\in Q(\boldsymbol{t}',k')$ and so $\sigma\langle u,v\rangle=1$.

*Case* 1.2: $v\in V(\widehat{CCC}_n)-F(\boldsymbol{t},k)$: $v\in P(\boldsymbol{t}',k')$ and so $\sigma\langle u,v\rangle=0$.

*Case* 2: $u\in Q(\boldsymbol{t},k)$: From Lemma 22, the vertices adjacent to $u$ are contained in $P(\boldsymbol{t},k)\cup\mathscr{Q}$.

*Case* 2.1: $v\in F(\boldsymbol{t},k)$: $v\notin Q(\boldsymbol{t},k)$ and so $\sigma\langle u,v\rangle=1$.

*Case* 2.2: $v\in V(\widehat{CCC}_n)-F(\boldsymbol{t},k)$: $v\in Q(\boldsymbol{t},k)$ and so $\sigma\langle u,v\rangle=0$.

Thus, neither (i) nor (ii) holds for any arc $\langle u,v\rangle$.

*7.2.4. Proof of Claim 16*

The claim follows from the fact that $Q(\boldsymbol{t},k)\cap F(\boldsymbol{t},k)=P(\boldsymbol{t},k)\cap F(\boldsymbol{t},1-k)=\emptyset$ for any $\boldsymbol{t}\in[2]^{n/2}$ and any $k\in[2]$.

*7.2.5. Proof of Claim 17*

**Lemma 23.** $|\mathscr{Q}| = 2^{n+2}$.

**Proof.** The lemma follows from the fact that $\mathscr{Q} = [2]^n \times \{0, \frac{n}{2} - 1, \frac{n}{2}, n-1\}$. $\square$

**Lemma 24.** *For any* $\boldsymbol{t} \in [2]^{n/2}$ *and* $k \in [2]$,

$$|P(\boldsymbol{t}, k)| = \left(\frac{n}{2} - 2\right) 2^{n/2}.$$

**Proof.** The lemma follows from the fact that

$$P(\boldsymbol{t}, 0) = \left\{[\boldsymbol{t} \cdot \boldsymbol{s}, i] : \ \boldsymbol{s} \in [2]^{n/2}, 1 \leqslant i \leqslant \frac{n}{2} - 2\right\}, \quad \text{and}$$

$$P(\boldsymbol{t}, 1) = \left\{[\boldsymbol{s} \cdot \boldsymbol{t}, i] : \ \boldsymbol{s} \in [2]^{n/2}, \frac{n}{2} + 1 \leqslant i \leqslant n - 2\right\}. \qquad \square$$

From Lemmas 23 and 24, we have

$$\begin{aligned} |F(\boldsymbol{t}, k)| &= |P(\boldsymbol{t}, k)| + |\mathscr{Q} - Q(\boldsymbol{t}, k)| < |P(\boldsymbol{t}, k)| + |\mathscr{Q}| \\ &= 2^{n+2} + \left(\frac{n}{2} - 2\right) 2^{n/2} = \frac{4N}{n} + \mathrm{o}\left(\frac{N}{n}\right). \end{aligned}$$

## 8. Shuffle-exchange graphs and de Bruijn graphs

For any $\boldsymbol{x} = x_{n-1}x_{n-2}\cdots x_0 \in [2]^n$, define that

$$\tau(\boldsymbol{x}) = x_{n-2}\cdots x_0 x_{n-1}.$$

The $n$-dimensional shuffle-exchange graph, denoted by $SE_n$, is the graph defined as follows:

$$V(SE_n) = [2]^n;$$

$$E(SE_n) = \{(\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{y} = \tau(\boldsymbol{x}) \text{ or } \boldsymbol{x} = \tau(\boldsymbol{y})\} \cup \{(\boldsymbol{x}, \boldsymbol{y}) : \ \boldsymbol{y} = \chi_0(\boldsymbol{x})\}.$$

The $n$-dimensional de Bruijn graph, denoted by $dB_n$, is the graph defined as follows:

$$V(dB_n) = [2]^n,$$

$$E(dB_n) = \{(\boldsymbol{x}, \boldsymbol{y}) : \ R_{n-1}(\boldsymbol{x}) = L_{n-1}(\boldsymbol{y}) \text{ or } L_{n-1}(\boldsymbol{x}) = R_{n-1}(\boldsymbol{y})\}.$$

Let $N = |V(SE_n)| = |V(dB_n)| = 2^n$. In this section, we prove the following bounds:

**Theorem 16.** $\delta(\widehat{SE}_n) = \Omega\left(\frac{N}{\log N}\right)$.

**Theorem 17.** $\delta(\widehat{dB}_n) = \mathrm{O}\left(\frac{N}{\log N}\right)$.

Notice that $\delta(\widehat{SE}_n) \leqslant \delta(\widehat{dB}_n)$ since $SE_n$ is a subgraph of $dB_n$ [2]. Thus, we have the following corollaries from Theorems 16 and 17.

**Corollary 6.** $\delta(\widehat{SE}_n) = \Theta\left(\frac{N}{\log N}\right)$.

**Corollary 7.** $\delta(\widehat{dB}_n) = \Theta\left(\frac{N}{\log N}\right)$.

*8.1. Proof of Theorem 16*

**Lemma 25.** *If* $t = \lfloor 2N/(3n+4) \rfloor$ *then*

$$\Upsilon_{SE_n}(t+1) \geqslant t.$$

**Proof.** Assume contrary that $\Upsilon_{SE_n}(t+1) < t$. Then, there exists some $F \subseteq V(SE_n)$ with $|F| = t$ such that every connected component of the subgraph $G$ of $SE_n$ induced by the vertices in $V(SE_n) - F$ has size $t$ or smaller.

For any ordered pair of vertices $\boldsymbol{x} = x_{n-1} \cdots x_0$ and $\boldsymbol{y} = y_{n-1} \cdots y_0$, let $W[\boldsymbol{x}, \boldsymbol{y}]$ be the following walk in $SE_n$ connecting $\boldsymbol{x}$ and $\boldsymbol{y}$:

$$\begin{array}{ccccccccc} \boldsymbol{x} & = & x_{n-1}x_{n-2}\cdots x_0 & \to & x_{n-2}\cdots x_0 x_{n-1} & \to & x_{n-2}\cdots x_0 y_{n-1} & \to & \cdots \\ & \to & x_0 y_{n-1}\cdots y_1 & \to & y_{n-1}\cdots y_1 x_0 & \to & y_{n-1}\cdots y_1 y_0 & = & \boldsymbol{y}. \end{array}$$

Define $\mathscr{W}(\mathbf{z}) = \{W[\boldsymbol{x}, \boldsymbol{y}] : \mathbf{z} \in V(W[\boldsymbol{x}, \boldsymbol{y}])\}$ for any $\mathbf{z} \in V(SE_n)$, and $\mathscr{W}(S) = \bigcup_{\mathbf{z}\in S} \mathscr{W}(\mathbf{z})$ for any $S \subseteq V(SE_n)$.

**Claim 18.** *For any* $\mathbf{z} \in V(SE_n)$,

$$|\mathscr{W}(\mathbf{z})| \leqslant \left(\frac{3n}{2} + 1\right) N.$$

*Moreover,*

$$|\mathscr{W}(F)| \leqslant t\left(\frac{3n}{2} + 1\right) N.$$

**Proof of Claim 18.** Let $\mathbf{z} = z_{n-1} z_{n-2} \cdots z_0$. Since

$$\mathscr{W}(\mathbf{z}) = \bigcup_{i=0}^{n-1} \{W[\boldsymbol{x}, \boldsymbol{y}] : L_{n-1}(\mathbf{z}) = R_i(\boldsymbol{x}) \cdot L_{n-i-1}(\boldsymbol{y}), z_0 = x_i \text{ or } y_i\}$$
$$\cup \{W[\boldsymbol{x}, \boldsymbol{y}] : \ \mathbf{z} = \boldsymbol{x}\},$$

we have

$$|\mathscr{W}(\mathbf{z})| \leqslant 3n \cdot 2^{n-1} + 2^n = \left(\frac{3n}{2} + 1\right) N,$$

and

$$|\mathscr{W}(F)| \leqslant t\left(\frac{3n}{2}+1\right)N. \qquad \square$$

By Lemma 19, we have

$$|\mathscr{W}(F)| > N^2 - \frac{2N}{3n+4}N = \frac{3n+2}{3n+4}N^2.$$

On the other hand, by Claim 18, we have

$$|\mathscr{W}(F)| \leqslant \frac{2N}{3n+4}\left(\frac{3n}{2}+1\right)N = \frac{3n+2}{3n+4}N^2,$$

which is a contradiction. Hence,

$$\Upsilon_{SE_n}(t+1) \geqslant t.$$

This completes the proof of Lemma 25. $\square$

From Theorem 4 and Lemma 25, we have Theorem 16.

### *8.2. Proof of Theorem 17*

#### *8.2.1. Partition of $V(dB_n)$*

The following lemma is proved in [12].

**Lemma 26** (*Schwabe [12]*). *Let n be a positive integer. For any positive integer $\Psi \leqslant n$, there exists a partition $(V_1, V_2, \ldots, V_{2^{n-\Psi}})$ of $V(\widehat{dB}_n)$ such that $|V_i| = 2^{\Psi}$ for any i, and*

$$\sum_{i \neq j} |\{\langle u, v\rangle \in A(\widehat{dB}_n) :\ u \in V_i, v \in V_j\}| = \mathrm{O}\left(\frac{2^n}{\Psi}\right).$$

Let $(V_1, \ldots, V_{2^{n-\Psi}})$ be a partition of $V(\widehat{dB}_n)$ satisfying the condition in Lemma 26. For any positive integer $i \leqslant 2^{n-\Psi}$, define $P(i)$ and $Q(i)$ as follows:

$$Q(i) = \{u \in V_i :\ \langle u, v\rangle \in A(\widehat{dB}_n) \text{ for some } v \in V(dB_n) - V_i\},$$

$$P(i) = V_i - Q(i).$$

It is easy to see that $(P(1), P(2), \ldots, P(2^{n-\Psi}), Q(1), Q(2), \ldots, Q(2^{n-\Psi}))$ is a partition of $V(dB_n)$. Let $\mathscr{P} = \bigcup_i P(i)$ and $\mathscr{Q} = \bigcup_i Q(i)$.

#### *8.2.2. Syndrome and fault sets*

The syndrome $\sigma_\Psi$ for $\widehat{dB}_n$ is defined as follows:

$$\sigma_\Psi\langle u, v\rangle = \begin{cases} 0 & \text{if } \begin{cases} 1. & u, v \in P(i) \text{ for some } i, \text{ or} \\ 2. & u, v \in Q(i) \text{ for some } i, \end{cases} \\ 1 & \text{otherwise.} \end{cases}$$

We define $2^{n-\Psi}$ fault sets as follows: For any positive integer $i \leqslant 2^{n-\Psi}$,

$$F(i) = P(i) \cup (\mathcal{Q} - Q(i)).$$

We prove Theorem 17 by showing the following claims:

**Claim 19.** *For any positive integer $i \leqslant 2^{n-\Psi}$, $F(i)$ is a consistent fault set for $\sigma_\Psi$.*

**Claim 20.** $\bigcap_{i=1}^{2^{n-\Psi}} F(i) = \emptyset$.

**Claim 21.** $|F(i)| = \mathrm{O}\left(\frac{N}{\log N}\right)$ *for any positive integer $i \leqslant k$.*

*8.2.3. Proof of Claim 19*

We will prove the claim by showing that neither (i) nor (ii) below holds for any positive integer $i \leqslant 2^{n-\Psi}$:

(i) $\sigma_\Psi\langle u, v\rangle = 0$ if $u \in V(\widehat{dB}_n) - F(i)$ and $v \in F(i)$,
(ii) $\sigma_\Psi\langle u, v\rangle = 1$ if $u, v \in V(\widehat{dB}_n) - F(i)$.

Let $F(i)$ be a fault set, $u \in V(\widehat{dB}_n) - F(i)$, and $\langle u, v\rangle \in A(\widehat{dB}_n)$.

*Case* 1: $u \in P(i')$ for some $i' \neq i$: The vertices adjacent to $u$ are contained in $P(i') \cup Q(i')$.

*Case* 1.1: $v \in F(i)$: $v \in Q(i')$ and so $\sigma_\Psi\langle u, v\rangle = 1$.

*Case* 1.2: $v \in V(\widehat{dB}_n) - F(i)$: $v \in P(i')$ and so $\sigma_\Psi\langle u, v\rangle = 0$.

*Case* 2: $u \in Q(i)$: The vertices adjacent to $u$ are contained in $P(i) \cup \mathcal{Q}$.

*Case* 2.1: $v \in F(i)$: $v \notin Q(i)$ and so $\sigma_\Psi\langle u, v\rangle = 1$.

*Case* 2.2: $v \in V(\widehat{dB}_n) - F(i)$: $v \in Q(i)$ and so $\sigma_\Psi\langle u, v\rangle = 0$.

Thus, neither (i) nor (ii) holds for any arc $\langle u, v\rangle$.

*8.2.4. Proof of Claim 20*

The claim follows from the fact that $Q(i) \cap F(i) = \emptyset$ for any $i$, and $P(i) \cap F(i') = \emptyset$ for any distinct $i$ and $i'$.

*8.2.5. Proof of Claim 21*

By Lemma 26, we have

$$|P(i)| < |V_i| = 2^\Psi \quad \text{and} \quad |\mathcal{Q}| = \mathrm{O}\left(\frac{2^n}{\Psi}\right), \quad \text{and thus}$$

$$|F(i)| = |P(i) \cup (\mathcal{Q} - Q(i))| < |P(i) \cup \mathcal{Q}| = \mathrm{O}\left(2^\Psi + \frac{2^n}{\Psi}\right).$$

If we choose $\Psi = n - \lceil \log n \rceil$, we have

$$|F(i)| = \mathrm{O}\left(\frac{2^n}{n}\right) = \mathrm{O}\left(\frac{N}{\log N}\right).$$

## 9. Concluding remarks

It should be noted that the upper bounds shown in the paper are proved in a unified manner. Our proofs are based on a graph partition problem described below. Let $G$ be an $N$-vertex graph. For any $X \subseteq V(G)$, let $\partial(X)$ denote the set of vertices in $X$ adjacent to vertices in $V(G) - X$. Our partition problem is to find a partition $(V_1, \ldots, V_{k(N)})$ of $V(G)$ such that

(1) $|V_i - \partial(V_i)| = \mathrm{O}(f(N))$ for any $i$,
(2) $\left|\bigcup_{i=1}^{k(N)} \partial(V_i)\right| = \sum_{i=1}^{k(N)} |\partial(V_i)| = \mathrm{O}(g(N))$, and
(3) $f(N) + g(N)$ is minimum.

Our proofs are based on the fact that if $f(N) + g(N) = \mathrm{O}(q(N))$ then the degree of sequential diagnosability of $G$ is $\mathrm{O}(q(N))$.

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