LETTER

A Note on the Complexity of Scheduling for Precedence Constrained Messages in Distributed Systems

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SUMMARY This note considers a problem of minimum length scheduling for a set of messages subject to precedence constraints for switching and communication networks, and shows some improvements upon previous results on the problem.

key words: *scheduling, NP-completeness, approximation algorithm*

1. Introduction

This note considers a problem of minimum length scheduling for a set of messages subject to precedence constraints for switching and communication networks. The problem was first studied by Barcaccia, Bonuccelli, and Di Iannii [1].

We consider a network with n inputs and n outputs. The messages to be sent are represented by an $n \times n$ matrix $D = [d_{ij}]$, the traffic matrix, whose entries are nonnegative integers, where $0 \le i, j \le n-1$. Entry d_{ij} represents the number of messages to be sent from input i to output j. In order to specify precedence constraints among messages, we represent a traffic matrix D by a sequence of $n \times n$ matrices $\mathbf{D} = (D^{(0)}, D^{(1)}, \ldots, D^{(k-1)})$ such that $D = \sum_{r=0}^{k-1} D^{(r)}$. We consider precedence constraints on the rows, which means that the entries in each row of $D^{(r+1)}$ can be scheduled only if the entries in the corresponding row of $D^{(r)}$ have already been scheduled $(0 \le r \le k-2)$.

A switching matrix is a binary matrix with at most one nonzero entry in each row and in each column. A switching matrix represents messages that can be sent simultaneously without conflicts.

A sequence of $n \times n$ switching matrices $\mathbf{S} = (S^{(0)}, S^{(1)}, \ldots, S^{(t-1)})$ is called a switching schedule for $\mathbf{D}$ if the following conditions are satisfied:

(1) $$\sum_{r=0}^{t-1} S^{(r)} = \sum_{r=0}^{k-1} D^{(r)} = D;$$

(2) For any integers p, $0 \le p \le k-1$, and i, $0 \le i \le n-1$, there exists an integer q, $0 \le q \le t-1$, such that $\sum_{r=0}^{q} s_{ij}^{(r)} = \sum_{r=0}^{p} d_{ij}^{(r)}$ holds for every j, $0 \le j \le n-1$.

Notice that condition (2) represents the precedence constraints on the rows. Integer t is called the length of $\mathbf{S}$ and denoted by $|\mathbf{S}|$.

We consider the following problems.

Problem 1 (PCRMS): Given $\mathbf{D} = (D^{(0)}, D^{(1)}, \ldots, D^{(k-1)})$ and positive integer h, decide if there exists a switching schedule $\mathbf{S}$ for $\mathbf{D}$ with $|\mathbf{S}| \le h$. □

Problem 2 (MIN-PCRMS-k): Given $\mathbf{D} = (D^{(0)}, D^{(1)}, \ldots, D^{(k-1)})$, find a switching schedule $\mathbf{S}$ for $\mathbf{D}$ with minimum length. □

It is shown in [1] that PCRMS is NP-complete if $k = 2$, $D^{(0)}$ is a binary matrix and $D^{(1)}$ is a ternary matrix, and $h = 3$. We improve this by showing the following.

Theorem 1: PCRMS is NP-complete if $k = 2$, $D^{(0)}$ and $D^{(1)}$ are binary matrices, and $h = 3$. □

It should be noted that PCRMS can be solved in polynomial time if $k = 1$ or $h \le 2$. In particular, MIN-PCRMS-1 can be solved in polynomial time by solving the edge coloring problem for a bipartite graph associated with $D^{(0)}$.

It follows from Theorem 1 that even MIN-PCRMS-2 is NP-hard. It is proved in [1] that for any positive integer k and positive number $\epsilon < 4/3$, there exists no polynomial time ϵ-approximation algorithm for MIN-PCRMS-k unless P = NP. It is also mentioned in [1] that the following naive algorithm is a polynomial time k-approximation algorithm for MIN-PCRMS-k.

Algorithm 1:

Step 1: Find an optimal switching schedule for $D^{(r)}$ $(0 \le r \le k-1)$.

Step 2: Schedule $D^{(r+1)}$ after the schedule for $D^{(r)}$ $(0 \le r \le k-2)$. □

Thus, the approximation ratio of a polynomial time approximation algorithm for MIN-PCRMS-k is between 4/3 and k if $k \ge 2$.

We show an estimate of the approximation ratio of Algorithm 1 by means of the structure of $\mathbf{D}$. For an $n \times n$ matrix $M = [m_{ij}]$, define that

$$L(M) = \max\left\{ \sum_{k=0}^{n-1} m_{ik}, \sum_{k=0}^{n-1} m_{kj} \middle| 0 \le i, j \le n-1 \right\},$$

$$l(M) = \min\left\{ \sum_{k=0}^{n-1} m_{ik}, \sum_{k=0}^{n-1} m_{kj} \middle| 0 \le i, j \le n-1 \right\}.$$

For $\mathbf{D} = (D^{(0)}, D^{(1)}, \ldots, D^{(k-1)})$, define that

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$$\alpha(\mathbf{D}) = \min\left\{\frac{l(D^{(r)})}{L(D^{(r)})}\middle| 0 \le r \le k-1\right\},$$

$$\beta(\mathbf{D}) = \max\left\{\frac{l(D^{(r)})}{L(D^{(r)})}\middle| 0 \le r \le k-1\right\}.$$

It should be noted that $L(D^{(r)})$ is the length of an optimal switching schedule for $D^{(r)}$ $(0 \le r \le k-1)$.

Theorem 2: The approximation ratio of Algorithm 1 for MIN-PCRMS-k is at most $2 - \beta(\mathbf{D})$ if $k = 2$, and at most $k - (k-1)\alpha(\mathbf{D})$ if $k \ge 3$. □

Theorem 3: The approximation ratio of Algorithm 1 for MIN-PCRMS-k is at least $k - (k-1)\beta(\mathbf{D})$ for any positive integer k. □

It follows from Theorems 2 and 3 that the approximation ratio of Algorithm 1 for MIN-PCRMS-2 is exactly $2 - \beta(\mathbf{D})$.

2. Proof of Theorem 1

We first need some preliminaries. Let $B = (X, Y, E)$ be a bipartite graph with maximum vertex degree 3, where (X, Y) is a bipartition of B, and E is the set of edges of B. We denote by X^δ and Y^δ the sets of vertices in X and Y with degree δ, respectively. Let E_1 be a perfect matching of B, and E_2 be a perfect matching of $(X', Y', E - E_1)$, where X' and Y' denote the sets of nonisolated vertices in X and Y, respectively, after the removal of the edges in E_1. (E_1, E_2) is called a double perfect matching for B. It is mentioned in [1] that the following problem is NP-complete:

Problem 3 (DPM-3): Given a bipartite graph $B = (X, Y, E)$ with maximum vertex degree 3, and $|X^\delta| = |Y^\delta|$ $(1 \le \delta \le 3)$, decide if there exists a double perfect matching for B. □

Now we are ready to prove the theorem. It is obvious that our problem is in NP. We prove the theorem by showing a polynomial time reduction from DPM-3 to PCRMS.

Let $B = (X, Y, E)$ be a bipartite graph as an instance of DPM-3. Let $n_\delta = |X^\delta| = |Y^\delta|$ $(1 \le \delta \le 3)$, and $X = \{x_0, \ldots, x_{n-1}\}$, $X^1 = \{x_0, \ldots, x_{n_1-1}\}$, $X^2 = \{x_{n_1}, \ldots, x_{n_1+n_2-1}\}$, $Y = \{y_0, \ldots, y_{n-1}\}$, $Y^1 = \{y_0, \ldots, y_{n_1-1}\}$, and $Y^2 = \{y_{n_1}, \ldots, y_{n_1+n_2-1}\}$. We assume without loss of generality that $n_1 \ne 1$. For any $F \subseteq X \times Y$, $M(F) = [m_{ij}]$ is an $n \times n$ binary matrix defined as:

$$m_{ij} = \begin{cases} 1 & \text{if } (x_i, y_j) \in F, \\ 0 & \text{otherwise.} \end{cases}$$

M is considered as a bijection from $2^{X \times Y}$ to the set of $n \times n$ binary matrices.

We define matrices $D^{(0)}$ and $D^{(1)}$ as follows: $D^{(0)} = M(E)$; $D^{(1)} = D'^{(1)} + D''^{(1)}$ where $D'^{(1)} = [d'^{(1)}_{ij}]$ and $D''^{(1)} = [d''^{(1)}_{ij}]$ are binary matrices defined as

$$d'^{(1)}_{ij} = \begin{cases} 1 & \text{if } j = i + 1 \le n_1 - 1 \text{ or } (i, j) = (n_1 - 1, 0), \\ 0 & \text{otherwise;} \end{cases}$$

$$d''^{(1)}_{ij} = \begin{cases} 1 & \text{if } i = j \le n_1 + n_2 - 1, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, $D^{(0)}$ and $D^{(1)}$ can be constructed in polynomial time. It is easy to see that $L(D^{(0)} + D^{(1)}) = l(D^{(0)} + D^{(1)}) = 3$.

We will prove that there exists a double perfect matching (E_1, E_2) for B if and only if there exists a switching schedule $\mathbf{S}$ for $\mathbf{D} = (D^{(0)}, D^{(1)})$ with $|\mathbf{S}| = 3$.

If there exists a double perfect matching (E_1, E_2) for B, then $(M(E_1), M(E_2) + D'^{(1)}, M(E - (E_1 \cup E_2)) + D''^{(1)})$ is a switching schedule for D with length 3.

Conversely, if there exists a switching schedule $\mathbf{S} = (S^{(0)}, S^{(1)}, S^{(2)})$ for $\mathbf{D}$, then $(M^{-1}(S^{(0)}), M^{-1}(QS^{(1)}))$ is a double perfect matching for B, where $Q = [q_{ij}]$ is an $n \times n$ binary matrix defined as

$$q_{ij} = \begin{cases} 1 & \text{if } i = j \ge n_1, \\ 0 & \text{otherwise.} \end{cases}$$

3. Proof of Theorem 2

Let $L_r = L(D^{(r)})$ and $l_r = l(D^{(r)})$, $0 \le r \le k-1$, and ρ_k be the approximation ratio of Algorithm 1 for MIN-PCRMS-k.

Lemma 1: $\rho_k \le \dfrac{\sum_{r=0}^{k-1} L_r}{\max\{L_r + \sum_{t \ne r} l_t \mid 0 \le r \le k-1\}}$.

Proof: Since L_r is the length of the optimal switching schedule for $D^{(r)}$ $(0 \le r \le k-1)$, $\sum_{r=0}^{k-1} L_r$ is the length of a switching schedule produced by Algorithm 1 for $\mathbf{D}$.

On the other hand, the length of the optimal switching schedule for $\mathbf{D}$ is at least

$$\max\left\{\sum_{r=0}^{k-1}\sum_{k=0}^{n-1} d^{(r)}_{ik}, \sum_{r=0}^{k-1}\sum_{k=0}^{n-1} d^{(r)}_{kj} \middle| 0 \le i, j \le n-1\right\}$$
$$\ge \max\left\{L_r + \sum_{t \ne r} l_t \middle| 0 \le r \le k-1\right\}.$$

Thus we have the lemma. □

We first consider the case when $k = 2$. Assume without loss of generality that $\beta(\mathbf{D}) = l(D^{(0)})/L(D^{(0)})$. We distinguish two cases.

(i) If $L_0 + l_1 \le l_0 + L_1$ then by Lemma 1 we have the following.

$$\begin{aligned} \rho_2 &\le \frac{L_0 + L_1}{l_0 + L_1} \\ &= 1 + \frac{L_0 - l_0}{l_0 + L_1} \\ &\le 1 + \frac{L_0 - l_0}{L_0 + l_1} \\ &\le 1 + \frac{L_0 - l_0}{L_0} \\ &= 2 - \beta(\mathbf{D}). \end{aligned}$$

(ii) If $L_0 + l_1 > l_0 + L_1$ then by Lemma 1 we have the following.

$$
\begin{aligned}
\rho_2 &\leq \frac{L_0 + L_1}{L_0 + l_1} \\
&= 1 + \frac{L_1 - l_1}{L_0 + l_1} \\
&< 1 + \frac{L_0 - l_0}{L_0 + l_1} \\
&\leq 1 + \frac{L_0 - l_0}{L_0} \\
&= 2 - \beta(\mathbf{D}).
\end{aligned}
$$

We next consider the case when $k \geq 3$. Assume without loss of generality that $\max\{L_r + \sum_{t \neq r} l_t \,|\, 0 \leq r \leq k-1\} = L_0 + \sum_{t=1}^{k-1} l_t$. It follows that $L_0 + l_t \geq l_0 + L_t$ for any $t \geq 1$. Thus by Lemma 1 we have the following.

$$
\begin{aligned}
\rho_k &\leq \frac{\sum_{r=0}^{k-1} L_r}{L_0 + \sum_{t=1}^{k-1} l_t} \\
&= 1 + \frac{\sum_{r=1}^{k-1}(L_r - l_r)}{L_0 + \sum_{t=1}^{k-1} l_t} \\
&\leq 1 + \frac{(k-1)(L_0 - l_0)}{L_0 + \sum_{t=1}^{k-1} l_t} \\
&\leq 1 + \frac{(k-1)(L_0 - l_0)}{L_0} \\
&= k - (k-1)\frac{l_0}{L_0} \\
&\leq k - (k-1)\alpha(\mathbf{D}).
\end{aligned}
$$

4. Proof of Theorem 3

Let $\mathbf{D} = (D^{(0)}, D^{(1)}, \ldots, D^{(k-1)})$ be a sequence of $n \times n$ matrices defined as:

$$
d_{ij}^{(0)} = \begin{cases} 1 & \text{if } i = j \text{ or } i = 0, \\ 0 & \text{otherwise;} \end{cases}
$$

$$
d_{ij}^{(r)} = \begin{cases} 1 & \text{if } i = r \text{ and } i \neq j, \\ 0 & \text{otherwise,} \end{cases}
$$

where $1 \leq r \leq k-1$.

It is obvious that $L(D^{(0)}) = n$, $l(D^{(0)}) = 1$, and $L(D^{(r)}) = n-1$, $l(D^{(r)}) = 0$ for $1 \leq r \leq k-1$. It follows that $\beta(\mathbf{D}) = 1/n$.

Since $L(D^{(0)}) = n$, and $L(D^{(r)}) = n-1$ for $1 \leq r \leq k-1$, the length of a switching schedule produced by Algorithm 1 for $\mathbf{D}$ is $n + (k-1)(n-1)$.

On the other hand, if we define a sequence of switching matrices $\mathbf{S} = (S^{(0)}, S^{(1)}, \ldots, S^{(n-1)})$ as:

$$
s_{ij}^{(r)} = \begin{cases} 1 & \text{if } j \equiv i + r \pmod{n}, \\ 0 & \text{otherwise,} \end{cases}
$$

where $0 \leq r \leq n-1$, then $\mathbf{S}$ is an optimal switching schedule for $\mathbf{D}$, since $L(D^{(0)}) = n$. Thus we have the following.

$$
\begin{aligned}
\rho_k &\geq \frac{n + (k-1)(n-1)}{n} \\
&= k - (k-1)\frac{1}{n} \\
&= k - (k-1)\beta(\mathbf{D}).
\end{aligned}
$$

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