On the Three-Dimensional Channel Routing

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***Abstract*— The 3-D channel routing is a fundamental problem on the physical design of 3-D integrated circuits. The 3-D channel is a 3-D grid G and the terminals are vertices of G located in the top and bottom layers. A net is a set of terminals to be connected. The object of the 3-D channel routing problem is to connect the terminals in each net with a tree (wire) in G using as few layers as possible and as short wires as possible in such a way that wires for distinct nets are disjoint. This paper shows that any set of n 2-terminal nets can be routed in a 3-D channel with $O\left(\sqrt{n}\right)$ layers using wires of length $O\left(\sqrt{n}\right)$. We also show that there exists a set of n 2-terminal nets that requires a 3-D channel with $\Omega\left(\sqrt{n}\right)$ layers to be routed.**

I. Introduction

The three-dimensional (3-D) integration is an emerging technology to implement large circuits, and currently being extensively investigated. (See [1]–[6], [8], for example.) In this paper, we consider a problem on the physical design of 3-D integrated circuits.

The 3-D channel routing is a fundamental problem on the physical design of 3-D integrated circuits. In the 3-D channel routing, the channel is a 3-D grid G consisting of columns, rows, and layers which are planes defined by fixing x-, y-, and z-coordinates, respectively. (See Fig. 1.) A terminal is a vertex of G located in the top or bottom layer. A net is a set of terminals to be connected. A net containing k terminals is called a k-net. A tree connecting the terminals in a net is called a wire. The object of the 3-D channel routing problem is to connect the terminals in each net with a wire in G using as few layers as possible and as short wires as possible in such a way that wires for distinct nets are disjoint. The number of layers is called the height of the 3-D channel. The purpose of this paper is to show the following two theorems.

Theorem 1: If the layers are square 2-D grids of area $4n$, the terminals are located on vertices with even x- and y-coordinates, and each net has terminals both in top and bottom layers, then any set of n 2-nets can be routed in a 3-D channel of height $O(\sqrt{n})$ using wires of length $O(\sqrt{n})$.

Theorem 2: There exists a set of n 2-nets that requires a 3-D channel of height $\Omega\left(\sqrt{n}\right)$ to be routed.

Theorem 1 implies that any set of n 2-nets can be routed in a 3-D channel of volume $O(n^{3/2})$. It should be noted that for the ordinary 2-D channel routing there exists a set of n 2-nets requiring a 2-D channel of area $\Omega(n^2)$ to be routed [7].

Other models for the 3-D channel routing can be found in the literature [3], [5], [8].

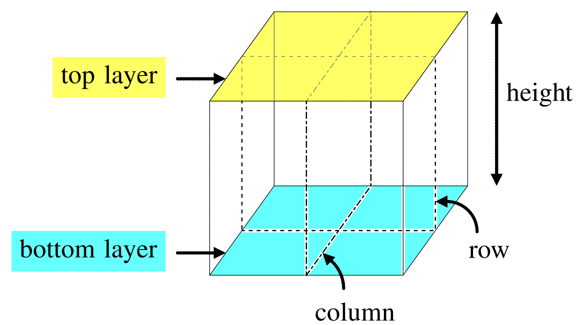

Fig. 1. The three-dimensional channel.

II. Preliminalies

We consider a 3-D channel of height $h+1$, which is a $2\sqrt{n} \times 2\sqrt{n} \times (h+1)$ 3-D grid. Each grid point is denoted by (x, y, z) with $0 \leq x, y \leq 2\sqrt{n}-1$ and $0 \leq z \leq h$. The column, row, and layer defined by $x = i$, $y = j$, and $z = k$ are called the i-column, j-row, and k-layer, respectively. The h-layer and 0-layer are corresponding to the top and bottom layers, respectively. Let $\mathcal{N} = \{N_i | 0 \leq i \leq n-1\}$ be a set of n 2-nets, and let $(X_i^{(h)}, Y_i^{(h)}, h)$ and $(X_i^{(0)}, Y_i^{(0)}, 0)$ be the terminals of N_i $(0 \leq i \leq n-1)$, where $X_i^{(h)}$, $Y_i^{(h)}$, $X_i^{(0)}$, and $Y_i^{(0)}$ are even, and $(X_i^{(h)}, Y_i^{(h)}, h) \neq (X_j^{(h)}, Y_j^{(h)}, h)$ and $(X_i^{(0)}, Y_i^{(0)}, 0) \neq (X_j^{(0)}, Y_j^{(0)}, 0)$ if $i \neq j$.

If $f : A \to B$ is a mapping, $f(A') = \{f(a) | a \in A'\}$ is the image of $A' \subseteq A$ and $f^{-1}(B') = \{a | f(a) = B'\}$ is the pre-image of $B' \subseteq B$. We denote by $f|_{A'}$ the restriction of f to A'. That is, $f|_{A'} : A' \to B$ and $f|_{A'}(a') = f(a')$ for $\forall a' \in A'$. If $g : B \to C$ is also a mapping, $g \circ f$ is a composite mapping from A to C defined as $g \circ f(a) = g(f(a))$ for $\forall a \in A$. A bijection $\pi : A \to A$ is called *a permutation on A*.

For a positive integer I, let $[I] = \{0, 1, \ldots, I-1\}$.

III. 2-D Channel Routing

We consider in this section a 2-D channel of height $m+1$, which is a $2m \times 2 \times (m+1)$ 3-D grid G'. Let $\mathcal{N}' = \{N_i' | i \in [m]\}$ be a set of m 2-nets, and let $(X_i^{(m)}, 0, m)$ and $(X_i^{(0)}, 0, 0)$ be the terminals of N_i' $(i \in [m])$, where $X_i^{(m)}$ and $X_i^{(0)}$ are even, and $X_i^{(m)} \neq X_j^{(m)}$ and $X_i^{(0)} \neq X_j^{(0)}$ if $i \neq j$.

Lemma 1: $\mathcal{N}'$ can be routed in G' so that no wire passes through the top layer.

Proof: Let $p_0, p_1, \ldots, p_k$ be grid points of G' such that p_i and p_{i+1} differ in just one coordinate, $i \in [k]$. Then, we denote

$$\left[\left(X_0^{(m)},0,m\right),\left(X_0^{(m)},0,m-1\right),\left(X_0^{(m)}+1,0,m-1\right),\left(X_0^{(m)}+1,0,0\right),\left(X_0^{(m)}+1,1,0\right),\left(X_0^{(0)},1,0\right),\left(X_0^{(0)},0,0\right)\right] \quad (1)$$

$$\left[\left(X_1^{(m)},0,m\right),\left(X_1^{(m)},0,1\right),\left(X_1^{(m)},1,1\right),\left(X_1^{(0)},1,1\right),\left(X_1^{(0)},0,1\right),\left(X_1^{(0)},0,0\right)\right] \quad (2)$$

$$\left[\left(X_i^{(m)},0,m\right),\left(X_i^{(m)},0,i\right),\left(X_i^{(m)},1,i\right),\left(X_i^{(0)}+1,1,i\right),\left(X_i^{(0)}+1,0,i\right),\left(X_i^{(0)}+1,0,0\right),\left(X_i^{(0)},0,0\right)\right] \quad (3)$$

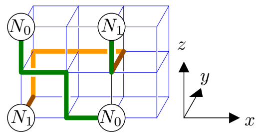

Fig. 2. A routing for a set of two 2-nets.

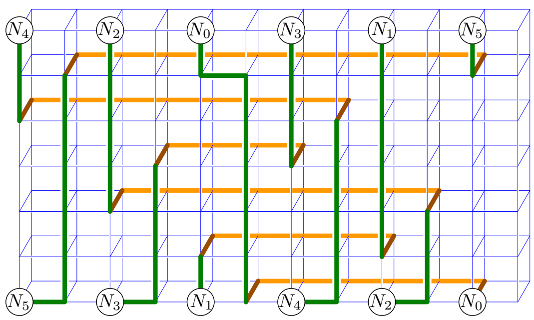

Fig. 3. A τ-routing for a set of six 2-nets.

by $[p_0, p_1, \ldots, p_k]$ a wire connecting p_0 and p_k obtained by connecting p_i and p_{i+1} by an axis-parallel line segment, $i \in [k]$. Suppose without loss of generality that $X_0^{(m)} = X_1^{(0)}$. Then, if $m \geq 3$, $\mathcal{N}'$ can be routed in G' using a wire defined by (1) for N_0', a wire defined by (2) for N_1', and wires defined by (3) for N_i', $2 \leq i \leq m$. It is not difficult to see that the wires defined above are disjoint. If $m = 2$, $\mathcal{N}'$ can be routed in G' as shown in Fig. 2. In either case, no wire passes through the top layer. ■

The routing defined in the proof of Lemma 1 is called *a τ-routing for $\mathcal{N}'$*. It is easy to see that a τ-routing can be computed in linear time. An example of τ-routing is shown in Fig. 3.

IV. Proof of Theorem 1

A. Technical Lemmas

For positive integers I and J, we define that $M = \{m_{i,j} | i \in [I], j \in [J]\}$, $M_{*j} = \{m_{i,j} | i \in [I]\}$, and $M_{i*} = \{m_{i,j} | j \in [J]\}$. Let D be a set with $|D| = J$ and $f : M \to D$ be a mapping such that

$$|f^{-1}(d)| \quad = \quad I \quad \text{for } \forall d \in D. \qquad (4)$$

Let π_j be a permutation on M_{*j} for $\forall j \in [J]$, and $\Pi = \{\pi_j | j \in [J]\}$. Define that $R_\Pi(i) = \bigcup_{j\in[J]} \pi_j^{-1}(m_{i,j})$. $|R_\Pi(i)| = J$, by definition. For such Π and each $i \in [I]$, we define that

$$\begin{aligned} W_\Pi(d,i) &= \begin{cases} 1 & \text{if } d \in f(R_\Pi(i)), \\ 0 & \text{if } d \notin f(R_\Pi(i)), \end{cases} \\ W_\Pi(i) &= \sum_{d\in D} W_\Pi(d,i), \text{ and} \\ W(\Pi) &= \sum_{i=0}^{I-1} W_\Pi(i). \end{aligned}$$

By definition, $1 \leq W_\Pi(i) \leq J$ and $W_\Pi(i) = J$ if and only if $f|_{R_\Pi(i)}$ is a bijection, that is, $|f(R_\Pi(i))| = J$.

Lemma 2: If $W_\Pi(i) \leq J - 1$, there exists $d \in D$ such that $\left|f|_{R_\Pi(i)}^{-1}(d)\right| \geq 2$, and there exists an integer $i' \in [I]$ suth that $d \notin f(R_\Pi(i'))$.

Proof: If $W_\Pi(i) \leq J - 1$ then $f|_{R_\Pi(i)}$ is not a bijection, and so $\left|f|_{R_\Pi(i)}^{-1}(d)\right| \geq 2$ for some $d \in D$ since $|D| = |R_\Pi(i)| = J$. It follows that $d \notin f(R_\Pi(i'))$ for some $i' \in [I]$ by (4). ■

We need the following easy lemma on directed multigraphs.

Lemma 3: For a directed multigraph H with the vertex set D, if there exists a vertex $d_0 \in D$ with $\deg_{\text{out}}(d_0) \geq \deg_{\text{in}}(d_0) + 1$ then there exist a vertex $d_p \in D$ such that $\deg_{\text{in}}(d_p) \geq \deg_{\text{out}}(d_p) + 1$ and a directed path $(d_0, d_1, \ldots, d_p)$ in H, where $\deg_{\text{in}}(d)$ and $\deg_{\text{out}}(d)$ is the in- and out-degrees of a vertex d in H.

Proof: Let $D_P \subseteq D$ be a set of vertices d' such that there exists a directed path from d_0 to d' in H, and let $H[D_P]$ be the induced subgraph of H on D_P. Let $\deg'_{\text{in}}(d')$ and $\deg'_{\text{out}}(d')$ be the in- and out-degrees of $d' \in D_P$ in $H[D_P]$, respectively. Notice that $\deg'_{\text{out}}(d') = \deg_{\text{out}}(d')$ and $\deg'_{\text{in}}(d') \leq \deg_{\text{in}}(d')$ for every $d' \in D_P$. Since $\deg'_{\text{out}}(d_0) = \deg_{\text{out}}(d_0) \geq \deg_{\text{in}}(d_0) + 1 \geq \deg'_{\text{in}}(d_0) + 1$, there exists a vertex $d' \in D_P$ such that $\deg'_{\text{out}}(d') \leq \deg'_{\text{in}}(d') - 1$, which follows from the fact that $\sum_{d'\in D_P} \deg_{\text{out}}(d') = \sum_{d'\in D_P} \deg_{\text{in}}(d')$. Since $\deg'_{\text{out}}(d') = \deg_{\text{out}}(d')$ and $\deg'_{\text{in}}(d') \leq \deg_{\text{in}}(d')$, we have $\deg_{\text{out}}(d') \leq \deg_{\text{in}}(d') - 1$. By the definition of D_P, there exists a directed path from d_0 to d'. ■

Lemma 4: There exists a set Π of permutations π_j on M_{*j} ($j \in [J]$) such that for every $i \in [I]$, $f|_{R_\Pi(i)} \circ \pi_j^{-1}(m_{i,j}) \neq f|_{R_\Pi(i)} \circ \pi_{j'}^{-1}(m_{i,j'})$ if $j \neq j'$.

Proof: By definition, $J \leq W(\Pi) \leq IJ = |M|$ for any set Π of permutations, and $W(\Pi) = |M|$ if and only if Π satisfies the condition in the lemma. In order to prove the lemma, it suffices to show the following.

Claim 1: Let Σ be a set of permutations σ_j on M_{*j} ($j \in [J]$) with $W(\Sigma) \leq IJ - 1$. Then, there exists a set Π of permutations π_j on M_{*j} ($j \in [J]$) such that $W(\Pi) \geq W(\Sigma) + 1$.

Proof of Claim 1: Since $W(\Sigma) \leq IJ - 1$, there exists $i_0 \in [I]$ such that $W_\Sigma(i_0) \leq J - 1$. By Lemma 2, there exist $d_0 \in D$ such that

$$\left| f|^{-1}_{R_\Sigma(i_0)}(d_0) \right| \geq 2 \tag{5}$$

and an integer $i_1 \in [I]$ such that $d_0 \notin f(R_\Sigma(i_1))$, i.e.,

$$\left| f|^{-1}_{R_\Sigma(i_1)}(d_0) \right| = 0. \tag{6}$$

Consider a directed multigraph H with vertex set D which has an arc $a_j = (f(\sigma_j^{-1}(m_{i_0,j})), f(\sigma_j^{-1}(m_{i_1,j})))$ for each $j \in [J]$. From (5) and (6), we have

$$\deg_{\text{out}}(d_0) \geq 2, \tag{7}$$

$$\deg_{\text{in}}(d_0) = 0, \tag{8}$$

respectively, where $\deg_{\text{in}}(d)$ and $\deg_{\text{out}}(d)$ is the in- and out-degrees of a vertex d in H, respectively. From Lemma 3, there exists a vertex $d_p \in D$ with

$$\deg_{\text{in}}(d_p) \geq \deg_{\text{out}}(d_p) + 1, \tag{9}$$

and there exists a directed path $P = (d_0, d_1, \ldots, d_p)$ in H. Let a_{j_l} be an arc (d_l, d_{l+1}) of P, $l \in [p]$. Notice that $f(\sigma_{j_l}^{-1}(m_{i_0,j_l})) = d_l$ and $f(\sigma_{j_l}^{-1}(m_{i_1,j_l})) = d_{l+1}$ for $\forall l \in [p]$. Therefore, $f(\sigma_{j_l}^{-1}(m_{i_1,j_l})) = f(\sigma_{j_{l+1}}^{-1}(m_{i_0,j_{l+1}}))$ for $\forall l \in [p]$. Let $\mathcal{J}_P = \{j_0, j_1, \ldots, j_{p-1}\}$, and $D_P = \{d_0, d_1, \ldots, d_p\}$. For each $j \in [J]$, define that

$$\rho_j(m_{i,j}) = \begin{cases} m_{i,j} & \text{if } i \notin \{i_0, i_1\} \text{ or } j \notin \mathcal{J}_P, \\ m_{i_1,j} & \text{if } i = i_0 \text{ and } j \in \mathcal{J}_P, \\ m_{i_0,j} & \text{if } i = i_1 \text{ and } j \in \mathcal{J}_P. \end{cases}$$

Let $\pi_j = \rho_j \circ \sigma_j$, and $\Pi = \{\pi_j | j \in [J]\}$. Then by definition,

$$\pi_j^{-1}(m_{i,j}) = \begin{cases} \sigma_j^{-1}(m_{i,j}) & \text{if } i \notin \{i_0, i_1\} \text{ or } j \notin \mathcal{J}_P, \\ \sigma_j^{-1}(m_{i_1,j}) & \text{if } i = i_0 \text{ and } j \in \mathcal{J}_P, \\ \sigma_j^{-1}(m_{i_0,j}) & \text{if } i = i_1 \text{ and } j \in \mathcal{J}_P, \end{cases} \tag{10}$$

since $\rho_j^{-1} = \rho_j$. Since $R_\Pi(i) = R_\Sigma(i)$ if $i \notin \{i_0, i_1\}$ by (10), we have $f(R_\Pi(i)) = f(R_\Sigma(i))$ if $i \notin \{i_0, i_1\}$. Thus, by the definition of $W_\Pi(d, i)$,

$$\sum_{d \in D} W_\Pi(d, i) = \sum_{d \in D} W_\Sigma(d, i) \text{ if } i \notin \{i_0, i_1\}. \tag{11}$$

Also, for $\forall j \notin \mathcal{J}_P$, $f(\pi_j^{-1}(m_{i,j})) = f(\sigma_j^{-1}(m_{i,j}))$ and so $W_\Pi(d, i) = W_\Sigma(d, i)$ for $\forall d \notin D_P$. Thus, we have

$$\sum_{d \in D - D_P} W_\Pi(d, i) = \sum_{d \in D - D_P} W_\Sigma(d, i). \tag{12}$$

For $\forall l \in [p]$, $W_\Sigma(d_l, i_0) = 1$, since $f(\sigma_{j_l}^{-1}(m_{i_0,j_l})) = d_l$. Thus, we have

$$\begin{aligned} \sum_{d \in D_P} W_\Sigma(d, i_0) &= \sum_{l=0}^{p-1} W_\Sigma(d_l, i_0) + W_\Sigma(d_p, i_0) \\ &= p + W_\Sigma(d_p, i_0). \end{aligned} \tag{13}$$

For $\forall l \in [p]$, $W_\Sigma(d_{l+1}, i_1) = 1$, since $f(\sigma_{j_l}^{-1}(m_{i_1,j_l})) = d_{l+1}$. On the other hand, $W_\Sigma(d_0, i_1) = 0$ from (6). Therefore,

$$\sum_{d \in D_P} W_\Sigma(d, i_1) = p. \tag{14}$$

For $\forall l \in [p]$, $W_\Pi(d_{l+1}, i_0) = 1$, since $f(\pi_{j_l}^{-1}(m_{i_1,j_l})) = f(\sigma_{j_l}^{-1}(m_{i_0,j_l})) = d_{l+1}$ by the definitions of π_j and ρ_j. By the definition of d_0, there is an integer $j \notin \mathcal{J}_P$ such that $f(\sigma_j^{-1}(m_{i_0,j})) = d_0$. Since $f(\sigma_j^{-1}(m_{i_0,j})) = f(\pi_j^{-1}(m_{i_0,j})) = d_0$ for such j, $W_\Pi(d_0, i_0) = 1$. Thus, we have

$$\begin{aligned} \sum_{d \in D_P} W_\Pi(d, i_0) &= W_\Pi(d_0, i_0) + \sum_{l \in [p]} W_\Pi(d_{l+1}, i_0) \\ &= p + 1. \end{aligned} \tag{15}$$

For $\forall l \in [p]$, $W_\Pi(d_l, i_1) = 1$ since $f(\pi_{j_l}^{-1}(m_{i_1,j_l})) = f(\sigma_{j_l}^{-1}(m_{i_0,j_l})) = d_l$, and we have

$$\begin{aligned} \sum_{d \in D_P} W_\Pi(d, i_1) &= \sum_{l \in [p]} W_\Pi(d_l, i_1) + W_\Pi(d_p, i_1) \\ &= p + W_\Pi(d_p, i_1). \end{aligned} \tag{16}$$

From (9), if $W_\Sigma(d_p, i_0) = 1$ then there exists $j \notin \mathcal{J}_P$ such that $f(\sigma_j^{-1}(m_{i_1,j})) = f(\pi_j(m_{i_1,j})) = d_p$. This implies that $W_\Pi(d_p, i_1) = 1$ if $W_\Sigma(d_p, i_0) = 1$, i.e.,

$$W_\Pi(d_p, i_1) \geq W_\Sigma(d_p, i_0). \tag{17}$$

From (12)–(17), we obtain

$$W_\Pi(i_0) + W_\Pi(i_1) \geq W_\Sigma(i_0) + W_\Sigma(i_1) + 1. \tag{18}$$

From (11) and (18), we have $W(\Pi) \geq W(\Sigma) + 1$. This completes the proof of the claim and the lemma. ∎

A set of permutations Π satisfying the condition in Lemma 4 is called *a set of shuffle permutations*. It is easy to see that a set of shuffle permutations can be obtained from a set of identity mappings $\mathrm{id}_{M_{*j}}$ on M_{*j}, $j \in [J]$, in $O(|M|^2)$ time.

B. 3-D Channel Routing Algorithm

Now, we are ready to show a polynomial time algorithm for computing a routing for $\mathcal{N}$ in a 3-D channel with height $3\sqrt{n} + 1$. We use two virtual terminals $(X_i^{(l)}, Y_i^{(l)}, l)$ and $(X_i^{(m)}, Y_i^{(m)}, m)$ for each net N_i such that $X_i^{(h)} = X_i^{(l)}$, $Y_i^{(l)} = Y_i^{(m)}$, and $X_i^{(m)} = X_i^{(0)}$, where $h = 3\sqrt{n}$, $l = 2\sqrt{n}$, and $m = \sqrt{n}$. In order to obtain such virtual terminals, we only need to determine $Y_i^{(l)} = Y_i^{(m)}$ for $\forall i \in [n]$ since $X_i^{(h)}$, $Y_i^{(h)}$, $X_i^{(0)}$, and $Y_i^{(0)}$ are given as the problem instance. The algorithm consists of three phases and each of which uses $\sqrt{n} + 1$ layers. For each net N_k, we connect $(X_k^{(h)}, Y_k^{(h)}, h)$ with $(X_k^{(l)}, Y_k^{(l)}, l)$ in the first phase, $(X_k^{(l)}, Y_k^{(l)}, l)$ with $(X_k^{(m)}, Y_k^{(m)}, m)$ in the second phase, and $(X_k^{(m)}, Y_k^{(m)}, m)$ with $(X_k^{(0)}, Y_k^{(0)}, 0)$ in the last phase. Each phase is performed by applying τ-routing for $\sqrt{n}$ 2-D channels of height $\sqrt{n} + 1$.

The virtual terminals can be computed in polynomial time as follows. Let $I = J = \sqrt{n}$ and let $M = \{m_{i,j} | i \in [I], j \in [J]\}$ be the set defined as $m_{i,j} = N_k$ if $j = X_k^{(h)}/2$ and $i =$

Input	$\mathcal{N} = \{N_k \mid k \in [n]\}$ with terminals $(X_k^{(0)}, Y_k^{(0)}, 0)$ and $(X_k^{(h)}, Y_k^{(h)}, h)$ for $\forall k \in [n]$.
Output	Routing for $\mathcal{N}$.
Step 0	**for** $\forall k \in [n]$, Compute virtual terminals $(X_k^{(l)}, Y_k^{(l)}, l)$ and $(X_k^{(m)}, Y_k^{(m)}, m)$.
Step 1	**for** $\forall j \in [\sqrt{n}]$, Apply τ-routing to connect $(X_k^{(h)}, Y_k^{(h)}, h)$ and $(X_k^{(l)}, Y_k^{(l)}, l)$ with $Y_k^{(h)} = Y_k^{(l)} = 2j$ in G_{*j}^l.
Step 2	**for** $\forall i \in [\sqrt{n}]$, Apply τ-routing to connect $(X_k^{(l)}, Y_k^{(l)}, l)$ and $(X_k^{(m)}, Y_k^{(m)}, m)$ with $X_k^{(l)} = X_k^{(m)} = 2i$ in G_{i*}^m.
Step 3	**for** $\forall j \in [\sqrt{n}]$, Apply τ-routing to connect $(X_k^{(m)}, Y_k^{(m)}, m)$ and $(X_k^{(0)}, Y_k^{(0)}, 0)$ with $Y_k^{(m)} = Y_k^{(0)} = 2j$ in G_{*j}^0.
Step 4	**for** $\forall k \in [n]$, Output a wire for N_k by concatenating three wires for N_k above.

Fig. 4. 3-D Channel Routing Algorithm

$Y_k^{(h)}/2$. Define that $D = \bigcup_{i\in[n]}\{X_i^{(0)}\} = \{0, 2, \ldots, 2\sqrt{n}-2\}$ and $f(N_k) = X_k^{(0)}$ for $\forall k \in [n]$. Since f satisfies (4), we can obtain in polynomial time a set Π of shuffle permutations π_j on M_{*j}, $j \in [J]$, by Lemma 4. For an integer i such that $m_{i,j} = \pi_j(N_k)$ for $j = X_k^{(h)}/2$, we define that $Y_k^{(l)} = 2i$ for any $k \in [n]$.

Let G_{*j}^r be a $2 \times 2\sqrt{n} \times (\sqrt{n}+1)$-subgrid induced by a set of grid points:

$$\left\{(x, y, z) \mid x \in \{2j, 2j+1\}, y \in \left[2\sqrt{n}\right], r \le z \le r + \sqrt{n}\right\},$$

and G_{i*}^r be a subgrid induced by a set of grid points:

$$\left\{(x, y, z) \mid x \in \left[2\sqrt{n}\right], y \in \{2i, 2i+1\}, r \le z \le r + \sqrt{n}\right\}.$$

Since π_j is a permutation on M_{*j}, $Y_{k_1}^{(l)} \neq Y_{k_2}^{(l)}$ if $X_{k_1}^{(h)} = X_{k_2}^{(h)}$. Therefore, for all N_k with $X_k^{(l)} = 2j$, $(X_k^{(h)}, Y_k^{(h)}, h)$ and $(X_k^{(l)}, Y_k^{(l)}, l)$ can be connected in G_{*j}^l by applying τ-routing, $j \in [\sqrt{n}]$, since $X_k^{(h)} = X_k^{(l)}$ for $\forall k \in [n]$.

Lemma 5: If $Y_{k_1}^{(l)} = Y_{k_2}^{(l)}$ then $X_{k_1}^{(0)} \neq X_{k_2}^{(0)}$.

Proof: Let $Y_{k_1}^{(l)} = Y_{k_2}^{(l)} = 2i$, and $Y_{k_1}^{(h)} = 2i_1$, $Y_{k_2}^{(h)} = 2i_2$, $X_{k_1}^{(h)} = 2j_1$, and $X_{k_2}^{(h)} = 2j_2$. Then, $m_{i,j_1} = \pi_{j_1}(m_{i_1,j_1})$ and $m_{i,j_2} = \pi_{j_2}(m_{i_2,j_2})$. Since π_{j_1} and π_{j_2} are permutations, $j_1 \neq j_2$. Since Π is a set of shuffle permutations, we have $X_{k_1}^{(0)} = f(m_{i_1,j_1}) = f \circ \pi_{j_1}^{-1}(m_{i,j_1}) \neq f \circ \pi_{j_2}^{-1}(m_{i,j_2}) = f(m_{i_2,j_2}) = X_{k_2}^{(0)}$. Thus, we have the lemma. ∎

Since $X_k^{(m)} = X_k^{(0)}$ for $k \in [n]$, we have $X_{k_1}^{(m)} \neq X_{k_2}^{(m)}$ if $Y_{k_1}^{(l)} = Y_{k_2}^{(l)}$ by Lemma 5. Therefore, for all N_k with $Y_k^{(l)} = Y_k^{(m)} = 2i$, $(X_k^{(l)}, Y_k^{(l)}, l)$ and $(X_k^{(m)}, Y_k^{(m)}, m)$ can be connected in G_{i*}^m by applying τ-routing, $i \in [\sqrt{n}]$, since $Y_k^{(l)} = Y_k^{(m)}$ for $\forall k \in [n]$.

Since $X_k^{(m)} = X_k^{(0)}$ for $\forall k \in [n]$, $Y_{k_1}^{(0)} \neq Y_{k_2}^{(0)}$ if $X_{k_1}^{(m)} = X_{k_2}^{(m)}$. Therefore, for all N_k with $X_k^{(l)} = 2j$, $(X_k^{(m)}, Y_k^{(m)}, m)$ and $(X_k^{(0)}, Y_k^{(0)}, 0)$ can be connected in G_{*j}^l by applying τ-routing, $j \in [\sqrt{n}]$, since $X_k^{(m)} = X_k^{(0)}$ for $\forall k \in [n]$.

A wire for each 2-net N_k in $\mathcal{N}$ can be obtained by concatenating three wires for N_k above connecting terminals and virtual terminals.

Our 3-D channel routing algorithm is shown in Fig. 4. Since each of Steps 1–3 uses $\sqrt{n}+1$ layers, and m- and l-layers are used in two steps, $\mathcal{N}$ is routed in a 3-D channel of height $3\sqrt{n}+1$. Since the length of every wire of a τ-routing is at most $3\sqrt{n} + O(1)$, the maximum wire length of our 3-D channel routing algorithm is at most $9\sqrt{n} + O(1)$. This completes the proof of Theorem 1.

It should be noted that the time complexity of our 3-D channel routing algorithm is $O(n^2)$, since Step 0 takes $O(n^2)$ time, and any other step takes $O(n)$ time as easily seen.

V. Proof of Theorem 2

Let $\mathcal{N} = \{N_i \mid 0 \le i \le n-1\}$ be a set of n 2-nets such that $X_i^{(h-1)} \le \sqrt{n} - 2$ and $X_i^{(0)} \ge \sqrt{n}$ if $i \le n/2$, and $X_i^{(h-1)} \ge \sqrt{n}$ and $X_i^{(0)} \le \sqrt{n} - 2$ if $i \ge n/2 + 1$. Consider an arbitrary routing of $\mathcal{N}$ on a 3-D channel G and let h be the height of G. Then a wire for every net in $\mathcal{N}$ must pass across the $(\sqrt{n} - 1)$-column. Since the area of every column is $2\sqrt{n}h$, we have $2h\sqrt{n} \ge |\mathcal{N}| = n$. Thus, $h = \Omega(\sqrt{n})$.

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