On VLSI Decompositions for d-ary de Bruijn Graphs (Extended Abstract)

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***Abstract*— A VLSI decomposition of a graph G is a collection of isomorphic vertex-disjoint subgraphs (called building blocks) of G which together span G. The efficiency of a VLSI decomposition is the fraction of the edges of G which are present in the subgraphs. This paper gives a necessary condition and a sufficient condition for a graph to be a building block for d-ary de Bruijn graphs. We also show building blocks for d-ary de Bruijn graphs with asymptotically optimal efficiency. Furthermore, we list the most efficient universal d-ary de Bruijn building blocks we know of.**

I. Introduction

A VLSI decomposition of a graph G is a collection of isomorphic vertex-disjoint subgraphs of G which together span G. A graph H isomorphic to the subgraphs comprising the decomposition is called a building block for G. The efficiency of H is the fraction of the edges of G which are present in the copies of H. If H is a building block for any graph in a family of graphs $\{G_n\}$, H is called a universal building block for $\{G_n\}$. Finding an efficient building block for G is corresponding to the design of an efficient single VLSI chip with the property that many identical copies of this chip could be wired together to form a circuit represented by G.

A couple of pioneering works on universal building blocks for binary de Bruijn graphs can be found in the literature[1-3,5-7]. These works were motivated by the need to construct large Viterbi decoders. Schwabe [7] showed that a subgraph of the nth order binary de Bruijn graph B_n^2 is a universal building block for $\{B_m^2 | m \geq n\}$ (universal binary de Bruijn building block of order n) with efficiency $1-O(1/n)$. He also showed that this is asymptotically optimal by proving that the efficiency of any universal binary de Bruijn building block of order n is at most $1-\Omega(1/n)$. It is conjectured by Dolinar, Ko, and McEliece [2,3] that the efficiency of an optimal universal binary de Bruijn building block of order n is asymptotically equal to $1-2/(n+1)$. While optimal universal binary de Bruijn building blocks of order n are known for $n \leq 4$, it remains open to find optimal universal binary de Bruijn building blocks of order n for larger values of n.

To solve this problem, some necessary conditions and relatively more restrictive sufficient conditions for a graph to be a universal binary de Bruijn building block have been developed, and based on these sufficient conditions some relatively efficient universal binary de Bruijn building blocks have been constructed [1-3,7].

Universal building blocks for d-ary de Bruijn graphs are first considered by Kopřiva and Tvrdík [4]. They showed that a subgraph of the nth order d-ary de Bruijn graph B_n^d is a universal building block for $\{B_m^d | m \geq n\}$ (universal d-ary de Bruijn building block of order n) with efficiency $1 - o(1)$. This paper shows that a subgraph of B_n^d is a universal d-ary de Bruijn building block of order n with efficiency $1 - O(d/n)$. We also show that the efficiency of any universal d-ary de Bruijn building block of order n is at most $1-\Omega(1/n)$. These results are natural generalization of those for binary de Bruijn graphs derived by Schwabe [7]. Moreover, we show that some necessary conditions and sufficient conditions for universal binary de Bruijn building blocks can be naturally generalized for d-ary de Bruijn graphs.

The rest of the paper is organized as follows. Some terminologies, definitions, and preliminary results are given in Section 2. Section 3 is a summary of previous results on binary de Bruijn graphs. Section 4 and 5 discuss necessary conditions and sufficient conditions for universal d-ary de Bruijn building blocks, respectively. Asymptotically optimal universal d-ary de Bruijn building blocks are shown in Section 6. Finally, in Section 7 we will list the most efficient universal d-ary de Bruijn building blocks we know of.

Due to space limitation, the proofs of all theorems are omitted in the extended abstract, and will appear in the final version of the paper. The proofs can be found in [5]

II. Preliminaries

A. d-ary Vectors

For any positive integer d, let $[d] = \{0, 1, \ldots, d-1\}$. We define three mappings L_n, R_n, and C_n from $[d]^n$ to $[d]^{n-1}$ as follows. If $x = (x_1, x_2, \ldots, x_n) \in [d]^n$ then

$$L_n(x) = (x_1, x_2, \ldots, x_{n-1}) \text{ and}$$
$$R_n(x) = (x_2, x_3, \ldots, x_n).$$

B. d-ary Strings and Covers

A d-ary vector $(x_1, x_2, \ldots, x_n) \in [d]^n$ can be viewed as a d-ary string $x_1 x_2 \cdots x_n$ in a natural way. Let $[d]^* = \bigcup_{n \geq 0} [d]^n$, where $[d]^0$ is the set of an empty string ε. The length of d-ary string x, denoted by $|x|$, is defined as n if $x \in [d]^n$. The cost $\gamma_d(x)$ of a d-ary string x is defined as $d^{-|x|}$. The cost $\gamma_d(S)$ of a set S of d-ary strings is defined as $\sum_{x \in S} \gamma_d(x)$.

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The concatenation of two d-ary strings x and y is denoted by $x \cdot y$. The concatenation of k x's is denoted by x^k. A d-ary string x is called a substring of a d-ary string y if there exist two d-ary strings w and z such that $y = w \cdot x \cdot z$. String w is called a prefix of y.

A d-ary string x covers a d-ary string y if x is a substring of y. A set $S \subset [d]^*$ of d-ary strings covers a d-ary string y if there exists a d-ary string $x \in S$ that covers y. $S \subset [d]^*$ is a cover of $T \subset [d]^*$ if S covers every $y \in T$. $S \subset [d]^*$ is irreducible if no $x \in S$ is covered by other $y \in S$.

C. Digraphs

Let G be a digraph (directed graph). We denote the vertex set and arc (directed edge) set of G by $V(G)$ and $A(G)$, respectively. An arc from vertex u to v is denoted by (u, v). Let $X = (x_0, x_1, \dots, x_k)$ be a sequence of $k+1$ vertices, and $Y = (y_1, y_2, \dots, y_k)$ be a sequence of k arcs. (X, Y) is called a path of length k if the following two conditions are satisfied:

1) $y_i = (x_i, x_{i-1})$ or $y_i = (x_{i-1}, x_i)$ for any i $(1 \le i \le k)$,
2) $x_i \neq x_j$ for any i and j with $0 \le i < j \le k$.

(X, Y) is also called a (x_0, x_k)-path. A path (X, Y) is called a cycle if $x_0 = x_k$. y_i is called a forward arc if $y_i = (x_{i-1}, x_i)$, and a backward arc otherwise. A dipath is a path with no backward arcs. A dipath (X, Y) is also denoted by X for short. If a path has f forward arcs and b backward arcs, we define the net length of the path to be $|f - b|$. A cycle is said to be balanced if the net length of the cycle is equal to 0. A cycle is said to be unbalanced if it is not balanced. For a cycle (X, Y), the maximum subnet length of (X, Y) is defined to be the maximum net length of a subpath of (X, Y). A digraph G is connected if there exists a (u, v)-path for any vertices $u, v \in V(G)$. It is easy to see the following.

Theorem I: [8] Let H be a connected digraph. Then H has no unbalanced cycle if and only if there exists a mapping $\rho : V(H) \to \mathbb{Z}$ such that $\rho(y) = \rho(x) + 1$ if $(x, y) \in A(H)$.

The mapping ρ above is called a rank function for H. A balanced cycle C is said to have property $\mathcal{D}$ if both

$$|\{x \in V(C)|\ \rho(x) = \min_{y \in V(C)} \rho(y)\}|$$

and

$$|\{x \in V(C)|\ \rho(x) = \max_{y \in V(C)} \rho(y)\}|$$

are even for any rank function ρ for C. A digraph G is graded of rank m if there is a rank function $\rho : V(G) \to [m+1]$. G is graded if G is graded of rank m for some m.

The cycle space of a digraph G is a vector space generated by the cycles of G. A basis of the cycle space is called a fundamental n-basis if the basis is consisting of fundamental cycles with respect to a spanning ditree such that the maximum subnet length of each fundamental cycle is at most n.

D. Building Blocks for Digraphs

A digraph H is called a building block for a digraph G if there exists a spanning subdigraph $\overline{H}$ of G that is a vertex-disjoint union of several copies of H. The efficiency of H as a building block for G, denoted by $\mathit{eff}(H : G)$, is defined to be $|A(\overline{H})|/|A(G)|$. It is easy to see the following theorem.

Theorem II: [3] If H is a building block for G then

$$\mathit{eff}(H : G) = \frac{|V(G)| \times |A(H)|}{|V(H)| \times |A(G)|}.$$

E. d-ary de Bruijn Graphs and Universal Building Blocks

The nth order d-ary de Bruijn graph B_n^d is a digraph defined as follows:

$$V(B_n^d) = [d]^n;$$
$$A(B_n^d) = \{(x, y) | x, y \in V(B_n^d), L_n(x) = R_n(y)\}.$$

It should be noted that $|V(B_n^d)| = d^n$ and $|A(B_n^d)| = d^{n+1}$ by definition. Each vertex is labeled with d-ary string of length n. The arcs from x to $0L_n(x), 1L_n(x), \dots, (d-1)L_n(x)$ is labeled with $0x, 1x, \dots, (d-1)x$, respectively. If S is a set of strings, $B_n^d(S)$ is defined to be the graph obtained from B_n^d by deleting all arcs whose labels have a prefix in S.

A universal d-ary de Bruijn building block of order n is a spanning subdigraph of B_n^d that is a building block for any B_m^d with $m \ge n$. It is shown in [3] and easily derived from Theorem II that if H is a universal d-ary de Bruijn building block of order n then

$$\mathit{eff}(H : B_m^d) = \frac{|A(H)|}{|A(B_n^d)|}$$

for all $m \ge n$. This common value which is independent of m is called the efficiency of H as a universal d-ary de Bruijn building block.

III. Previous Results for Binary de Bruijn Graphs

A. Necessary Conditions

Theorem III: [3] If H is a universal binary de Bruijn building block then H does not contain an unbalanced cycle. That is, if H is a universal binary de Bruijn building block then H is graded.

Theorem IV: [3] If H is a universal binary de Bruijn building block then H does not contain two vertices x and y such that there are two dipaths of the same length from x to y.

Theorem V: [8] If H is a universal binary de Bruijn building block then every cycle of H has property $\mathcal{D}$.

It should be noted that Theorem V implies Theorem IV. That is, the necessary condition in Theorem V is more restrictive than that in Theorem IV.

B. Sufficient Conditions

Theorem VI: [1] If S is an irreducible cover of $[2]^n$ then $B_n^2(S)$ is a universal binary de Bruijn building block of order n with efficiency $1 - \gamma_2(S)$.

Theorem VII: [3] If a spanning subdigraph H of B_n^2 is graded of rank n then H is a universal binary de Bruijn building block of order n.

Theorem VIII: [8] Let H be a connected spanning subdigraph of B_n^2. If H is graded and has a fundamental $(n+1)$-basis of the cycle space, and every cycle of H has property

$\mathcal{D}$ then H is a universal binary de Bruijn building block of order n.

Since $B_n^2(S)$ is graded of rank n as easily verified, Theorem VI is immediate from Theorem VII. It should be noted that Theorem VIII implies Theorem VII. That is, the sufficient condition in Theorem VIII is less restrictive than that in Theorem VII.

C. Asymptotically Optimal Universal Building Blocks

Let S_S be the set of all binary strings of length $n/2$ such that one of the longest runs of zeroes in X is at either the beginning or the end of X. It can be shown that S_S is an irreducible cover of $[2]^n$ with $\gamma_2(S_S) = O(1/n)$.

Theorem IX: [7] $B_n^2(S_S)$ is a universal binary de Bruijn building block of order n with efficiency $1 - O(1/n)$.

S_S is asymptotically optimal since the following holds.

Theorem X: [7] The efficiency of a universal binary de Bruijn building block of order n is at most $1 - \Omega(1/n)$.

IV. Necessary Conditions

We can show that Theorems III and IV have natural generalizations for d-ary de Bruijn graphs as follows:

Theorem 1: If H is a universal d-ary de Bruijn building block then H is graded.

Theorem 2: If H is a universal d-ary de Bruijn building block then H does not contain two vertices x and y such that there are two dipaths of the same length from x to y.

We do not know whether Theorem V can be generalized for d-ary de Bruijn graphs.

V. Sufficient Conditions

It was shown in [4] that Theorem VI can be naturally generalized for d-ary de Bruijn graphs.

Theorem XI: [4] If S is an irreducible cover of $[d]^n$ then $B_n^d(S)$ is a universal d-ary de Bruijn building block of order n with efficiency $1 - \gamma_d(S)$.

We can also generalize Theorem VII. In fact, we can show a stronger theorem than the generalization of Theorem VII as follows.

Theorem 3: Let H be a connected spanning subdigraph of B_n^d. If H is graded and has a fundamental n-basis of the cycle space then H is a universal d-ary de Bruijn building block of order n.

Since $B_n^d(S)$ is graded of rank n, Theorem XI is immediate from Theorem 3. We do not know whether Theorem VIII can be generalized for d-ary de Bruijn graphs.

VI. Asymptotically Optimal Universal Building Blocks

An irreducible cover S_{KT} of $[d]^n$ with $\gamma_d(S_{KT}) = o(1)$ was shown in [4].

Theorem XII: [4] $B_n^d(S_{KT})$ is a universal d-ary de Bruijn building block of order n with efficiency $1 - o(1)$.

We show an irreducible cover of $[d]^n$ with asymptotically optimal cost. More precisely, we show that Theorems IX and X can be naturally generalized for d-ary de Bruijn graphs.

Let S_n^d and T_n^d be the set of all d-ary strings of length $n/2$ such that one of the longest runs of minimum values in w is at the beginning and the end of w, respectively. We can prove that $S_n^d \cup T_n^d$ is an irreducible cover of $[d]^n$ with $\gamma_d(S_n^d \cup T_n^d) = O(d/n)$, and the following.

Theorem 4: $B_n^d(S_n^d \cup T_n^d)$ is a universal d-ary de Bruijn building block of order n with efficiency $1 - O(d/n)$.

We can also show an upper bound for the efficiency as follows.

Theorem 5: The efficiency of a universal d-ary de Bruijn building block of order n is at most $1 - \Omega(1/n)$.

It should be noted that the efficiency of $B_n^d(S_n^d \cup T_n^d)$ is asymptotically optimal if d is fixed. It is open to close the gap between upper and lower bounds above.

VII. The Most Efficient Known Universal d-ary de Bruijn Building Blocks

We used simulated annealing to find an efficient universal d-ary de Bruijn building block of order n. A configuration is a spanning ditree of B_n^d. The neighborhood of a spanning tree T is the spanning ditrees T' such that $|A(T') - A(T)| = 1$. For a spanning ditree T of B_n^d, let H_T be a unique maximal universal d-ary de Bruijn building block of order n obtained from T by adding arcs of B_n^d in such a way that H_T satisfies the condition of Theorem VIII if $d = 2$, and the condition of Theorem 3 if $d \geq 3$. The cost of T is defined as $1 - \mathit{eff}(H_T : B_n^d)$.

Table 1 lists the number of arcs and efficiency of the most efficient H_T we have been able to find using simulated annealing, together with the efficiency obtained so far in the literature, [2] for $d = 2$ and [4] for $d \geq 3$. In the table a_n and e_n denote the number of arcs and the efficiency of H_T, respectively.

The best known universal ternary de Bruijn building block of order 5 is shown in Fig. 1, in which the ternary strings are represented by their decimal equivalents, and all edges are directed from left to right.

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TABLE I

THE MOST EFFICIENT KNOWN UNIVERSAL d-ARY DE BRUIJN BUILDING BLOCKS.

$n \backslash d$	2			3			4			5			6		
	a_n	e_n	[2]	a_n	e_n	[4]	a_n	e_n	[4]	a_n	e_n	[4]	a_n	e_n	[4]
1	1	0.250	0.250	2	0.222	0.000	4	0.250	0.000	6	0.240	0.000	9	0.250	0.000
2	3	0.375	0.375	11	0.407	0.222	27	0.422	0.188	53	0.424	0.160	92	0.426	0.139
3	8	0.500	0.500	44	0.543	0.296	140	0.547	0.234	343	0.549	0.192	703	0.542	0.162
4	19	0.594	0.594	153	0.630	0.395	647	0.632	0.340	1946	0.623	0.294	4840	0.622	0.258
5	43	0.672	0.672	499	0.684	0.519	2790	0.681	0.457	10476	0.670	0.403	–	–	0.359
6	92	0.719	0.719	1584	0.724	0.604	11707	0.715	0.546	–	–	0.490	–	–	0.442
7	193	0.754	0.754	4954	0.755	0.661	–	–	0.612	–	–	0.560	–	–	0.512
8	399	0.779	0.777	–	–	0.700	–	–	0.662	–	–	0.616	–	–	0.570
9	819	0.800	–	–	–	0.726	–	–	0.700	–	–	0.661	–	–	0.619
10	1673	0.817	–	–	–	0.743	–	–	0.728	–	–	0.697	–	–	0.659
11	3412	0.833	–	–	–	0.755	–	–	0.749	–	–	0.725	–	–	0.693

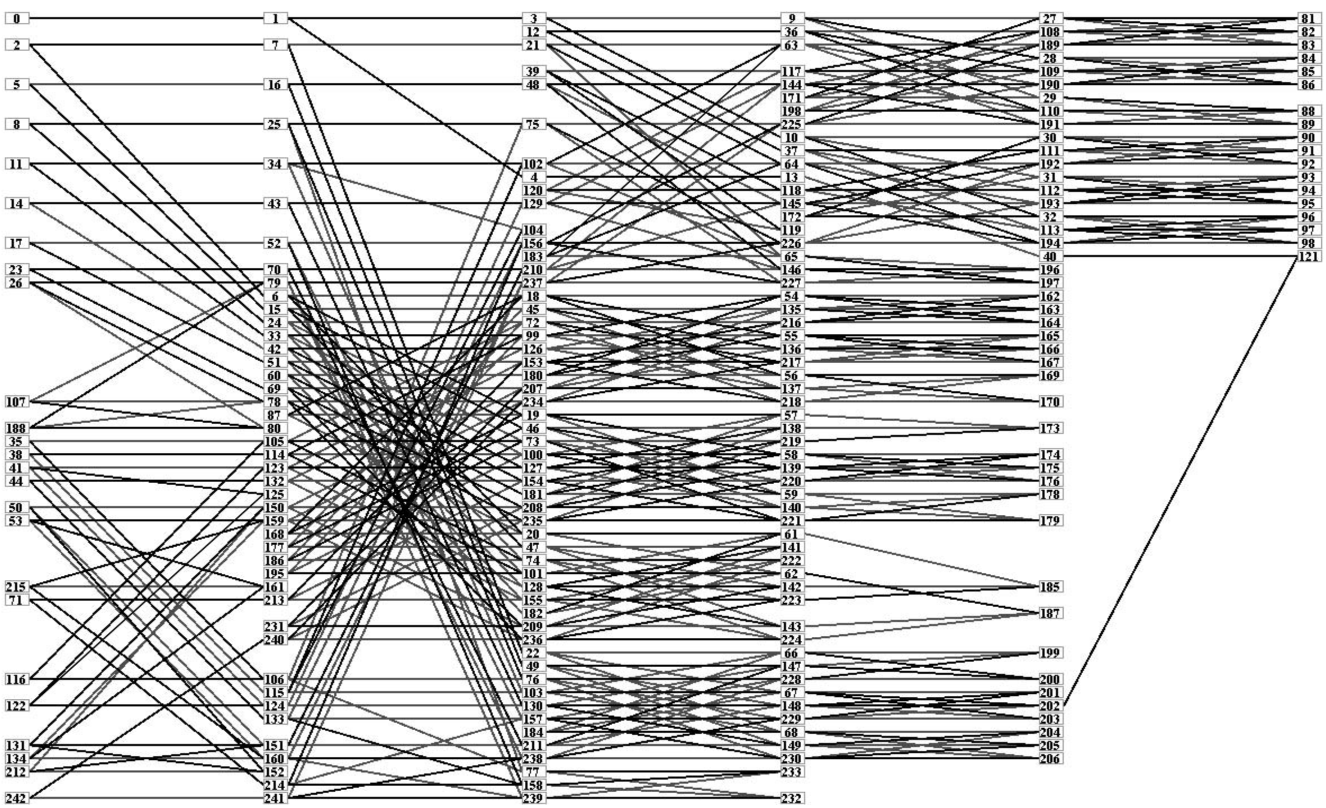

Fig. 1. The most efficient known universal ternary de Bruijn building block of order 5.