# On the Three-Dimensional Layout of Butterfly Networks

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Butterfly networks have served as interconnection networks of parallel computers and ATM switches, and have been extensively studied in the literature. This paper shows an efficient 3-D layout of the butterfly network.

The $n$-D *butterfly network* $B(n)$ is the graph defined as follows: $V(B(n)) = \{0, 1, \ldots, n\} \times \{0, 1\}^n$; $E(B(n)) = \{(\langle l, b\rangle, \langle l+1, b'\rangle) | 0 \le l \le n-1, b = b'$, or $b$ and $b'$ differ in precisely the $(l+1)$-th bit$\}$. For each $0 \le l \le n$, the $N = 2^n$ vertices in the set $\{l\} \times \{0, 1\}^n$ comprise level $l$ of $B(n)$. The vertices at level 0 are called *inputs*, and those at level $n$ are called *outputs*. $B(n)$ is also called an $N$*-input butterfly network*, and denoted by $B_N$. $B(3) = B_8$ is shown in Fig. 1. The 3-D $m_1 \times m_2 \times m_3$ *grid* $R(m_1, m_2, m_3)$ is the graph defined as follows: $V(R(m_1, m_2, m_3)) = \{0, 1, \ldots, m_1 - 1\} \times \{0, 1, \ldots, m_2 - 1\} \times \{0, 1, \ldots, m_3 - 1\}$; $E(R(m_1, m_2, m_3)) = \{(\langle u_1, u_2, u_3\rangle, \langle v_1, v_2, v_3\rangle) | \sum_{i=1}^{3} |u_i - v_i| = 1\}$.

An *embedding* $\langle \phi, \rho\rangle$ of a graph $G$ into a graph $H$ consists of a one-to-one mapping $\phi : V(G) \to V(H)$, together with a routing $\rho$ that maps each edge $(u, v) \in E(G)$ onto a path $\rho(u, v)$ in $H$ that connects vertices $\phi(u)$ and $\phi(v)$. An embedding $\langle \phi, \rho\rangle$ of a graph $G$ into a 3-D grid $R$ is called a 3-D *layout* of $G$ if routing paths $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint for any distinct $e_1, e_2 \in E(G)$. $|V(R)|$ is called the *volume* of the 3-D layout, and $\mathrm{vol}(G)$ is the minimum volume of a 3-D layout of $G$. The *bisection width* $\mathrm{bw}(G)$ of a graph $G$ is the minimum number of edges that must be removed from $G$ in order to partition $G$ into two equal-sized subgraphs to within one vertex. It is well-known that $\mathrm{vol}(G) \ge \mathrm{bw}(G)^{3/2}$ for any graph $G$ [3].

It is also well-known that $\mathrm{vol}(B_N) = \Theta(N^{3/2})$. In fact,

$$\mathrm{vol}(B_N) \quad \ge \quad 0.754N^{3/2} + o(N^{3/2}), \qquad (1)$$

since $\mathrm{bw}(B_N) = 2(\sqrt{2} - 1)N + o(N)$ [1]. On the other hand, the best previous upper bound is as follows:

$$\mathrm{vol}(B_N) \quad \le \quad 723\sqrt{2}N^{3/2} + o(N^{3/2}), \qquad (2)$$

which is derived from a result on the embedding of $B_N$ into a 3-D grid with edge-disjoint routing [3, 4] by a naive modification of replacing each vertex of the 3-D grid by a copy of $R(3, 3, 3)$ to make routing paths internally disjoint.

This paper shows that

$$\mathrm{vol}(B_N) \quad \le \quad 8\sqrt{2}N^{3/2} + o(N^{3/2}), \qquad (3)$$

which is a considerable improvement on (2). Our upper bound is obtained by an explicit 3-D layout of $B_N$, which is a modification of that introduced in [2].

Let $n_1 = \lceil (n-1)/2 \rceil$ and $n_2 = \lfloor (n-1)/2 \rfloor$. Then, it is easy to see that $B(n)$ can be decomposed into $2^{n_2+1}$ copies of $B(n_1)$ consisting of the vertices at levels 0 through $n_1$, and $2^{n_1+1}$ copies of $B(n_2)$ consisting of the vertices at levels $n_1 + 1$ through $n$ by deleting every edge connecting a vertex at level $n_1$ and a vertex at level $n_1 + 1$. For example, $B(7)$ can be decomposed into $2^{3+1} + 2^{3+1}$ copies of $B(3)$. Let $S(n)$ be the graph obtained from $B(n)$ by adding two new pendant vertices adjacent to each output. $S(3)$ is shown in Fig. 2. It is easy to see that $S(n)$ can be decomposed into edge-disjoint complete binary trees of height at most $n + 1$. We can prove that $S(n)$ can be laid out in $R(2^{\lfloor (n+1)/2 \rfloor + 1}, 2^{\lceil (n+1)/2 \rceil + 1}, 2^{\lfloor n/2 \rfloor - 1} + 1)$ using the $H$-tree layout for complete binary trees. Such a layout is called a *cube-layout* of $S(n)$, and denoted by $\Gamma(n)$. $\Gamma(3)$ is shown in Fig. 3.

The outline of our layout algorithm can be described as follows:

**1)** Decompose $B(n)$ into $2^{n_2+1}$ copies of $B(n_1)$ and $2^{n_1+1}$ copies of $B(n_2)$.

**2)** Pack $2^{n_2+1}$ upside-down copies of $\Gamma(n_1)$ into $G_1 = R(2^{n_1+1}, 2^{n_2+1}, 2^{n_2-1} + 1)$, and $2^{n_1+1}$ copies of $\Gamma(n_2)$ into $G_3 = R(2^{n_1+1}, 2^{n_2+1}, 2^{n_1-1} + 1)$.

**3)** Put $G_1$, $G_2 = R(2^{n_1+1}, 2^{n_2+1}, 2^{n_1} + 2^{n_2+1})$ and $G_3$ one atop another in this order, and connect the pairs of corresponding pendant vertices by internally disjoint paths in $G_2$. (See Fig. 4.)

It is easy to see that the volume of the resulting layout is $8\sqrt{2}N^{3/2} + o(N^{3/2})$.

It is open to close the gap between the bounds in (1) and (3).

## References

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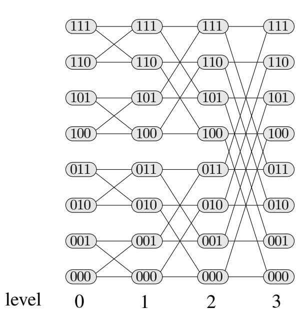


**Fig. 1** $B(3)$.

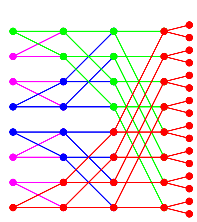

**Fig. 2** $S(3)$.

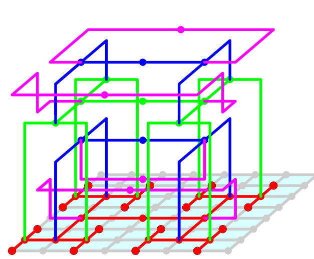

**Fig. 3** $\Gamma(3)$.

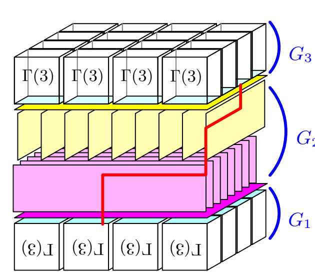


**Fig. 4** 3-D layout of $B(7)$.