LETTER

A Note on the Implementation of de Bruijn Networks by the Optical Transpose Interconnection System

Kohsuke OGATA[†], ***Nonmember***, **Toshinori YAMADA**[††], ***and*** **Shuichi UENO**[†a)], ***Members***

SUMMARY This note shows an efficient implementation of de Bruijn networks by the Optical Transpose Interconnection System (OTIS) extending previous results by Coudert, Ferreira, and Perennes [2].
key words: *interconnection networks, optical networks, optical transpose interconnection system (OTIS), de Bruijn networks*

1. Introduction

The OTIS architecture was proposed by Marsden, Marchand, Harvey, and Esener [1] to implement networks by free space optical interconnections. This architecture consists of the input and output arrays, and a pair of lenslet arrays between I/O arrays. The input array consists of p groups of q transmitters, and the output array consists of q groups of p receivers. The lenslet arrays consists of $p + q$ lenses. This architecture connects transmitter (i, j) to receiver $(q - j - 1, p - i - 1)$, $0 \leq i \leq p - 1, 0 \leq j \leq q - 1$.

The OTIS architecture is represented by a bipartite digraph $OTIS(p, q)$ defined as follows. Let $V(G)$ and $A(G)$ denote the vertex set and arc set of a digraph G, respectively. An arc from a vertex u to v is denoted by $[u, v]$. We define that $Z_n = \{0, 1, \ldots, n - 1\}$ for any positive integer n. $OTIS(p, q)$ is a bipartite digraph defined as:

$$V(OTIS(p,q)) = \{(i,j)_t | i \in Z_p, j \in Z_q\} \cup \{(k,l)_r | k \in Z_q, l \in Z_p\};$$

$$A(OTIS(p,q)) = \{[(i,j)_t, (k,l)_r] | i + l = p - 1, j + k = q - 1\}.$$

Let $H(p, q, d)$ be a d-regular digraph obtained from $OTIS(p, q)$ as follows:

$$V(H(p,q,d)) = Z_{\lceil pq/d \rceil};$$

$$A(H(p,q,d)) = \left\{ [u,v] \;\middle|\; \begin{array}{l} [(i,j)_t, (k,l)_r] \in A(OTIS(p,q)), \\ u = \lfloor (iq+j)/d \rfloor, v = \lfloor (kp+l)/d \rfloor \end{array} \right\}.$$

An OTIS architecture with 2 + 8 lenses shown in Fig. 1 is represented by $OTIS(2, 8)$ shown in Fig. 2. $H(2, 8, 2)$ shown in Fig. 3 is a 2-regular digraph obtained from $OTIS(2, 8)$. $OTIS(p, q)$ is called an OTIS layout for a d-regular digraph G if $H(p, q, d)$ is isomorphic to G. Our problem is to find an OTIS layout for a given digraph such that the number of lenses $p + q$ is as small as possible.

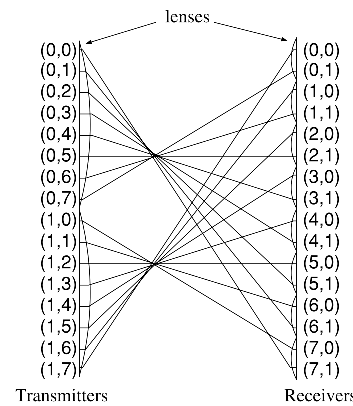

Fig. 1 OTIS architecture with 2 + 8 lenses.

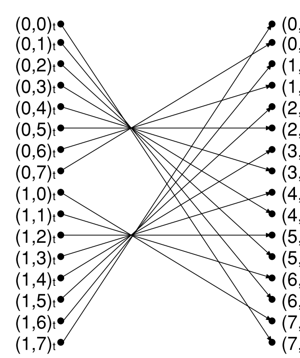

Fig. 2 $OTIS(2, 8)$.

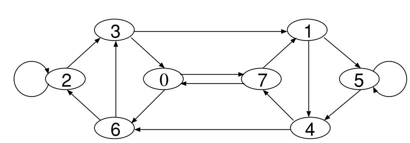

Fig. 3 $H(2, 8, 2)$.

The de Bruijn network has been extensively studied in connection with parallel and distributed computing. For any integer $d \geq 2$, the d-ary de Bruijn network of dimension D, denoted by $B(d, D)$, is a d-regular digraph defined as follows:

Manuscript received June 8, 2005.
Final manuscript received August 17, 2005.
[†]The authors are with the Dept. of Communications and Integrated Systems, Graduate School of Science and Engineering, Tokyo Institute of Technology, Tokyo, 152-8550 Japan.
[††]The author is with the Dept. of Information and Computer Sciences, Saitama University, Saitama-shi, 338-8570 Japan.
a) E-mail: ueno@lab.ss.titech.ac.jp
DOI: 10.1093/ietfec/e88–a.12.3661

Copyright © 2005 The Institute of Electronics, Information and Communication Engineers

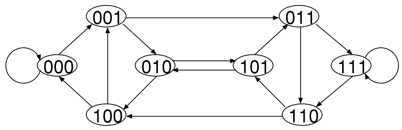

Fig. 4 $B(2,3)$.

$$V(B(d,D)) = Z_d^D;$$
$$A(B(d,D)) = \left\{ [u,v] \;\middle|\; \begin{array}{l} u,v \in V(B(d,D)) \\ u_i = v_{i+1} \quad for \quad \forall i \in Z_{D-1} \end{array} \right\},$$

where a vertex $x \in V(B(d,D))$ is denoted by $(x_0, x_1, \ldots, x_{D-1})$. $B(2,3)$ is shown in Fig. 4. Since $B(2,3)$ is isomorphic to $H(2,8,2)$, $OTIS(2,8)$ is an OTIS layout for $B(2,3)$.

Coudert, Ferreira, and Perennes [2] show that if D is even, $B(d,D)$ has an efficient OTIS layout with $O(d\sqrt{N})$ lenses, where $N = d^D = |V(B(d,D))|$. It should be noted that the number of lenses of an OTIS layout for an N-vertex d-regular digraph is $\Omega(\sqrt{dN})$ if any, as can be easily seen [2].

We show a natural extension to the result above by proving that for any positive integer D, $B(d,D)$ has an efficient OTIS layout. The number of lenses of the OTIS layout is $O(d^{\frac{5}{2}}\sqrt{N})$ if $D \equiv 1 \pmod 4$, $O(d^{\frac{3}{2}}\sqrt{N})$ if $D \equiv 3 \pmod 4$, and $O(d\sqrt{N})$ if D is even.

2. Main Results

Theorem 1: For any positive integers p' and q', $H(d^{p'}, d^{q'}, d)$ is isomorphic to $B(d, p'+q'-1)$ if and only if p' and q' are relatively prime.

Proof: We denote $H(d^{p'}, d^{q'}, d)$ by H, and $p'+q'-1$ by D. It is easy to see that $H(d^{p'}, d^{q'}, d)$ is not isomorphic to $B(d, p'+q'-1)$ if $p'q' = 0$. So we assume that $p'q' \neq 0$. A permutation on Z_n is a bijective mapping on Z_n. We define a permutation f_D on Z_D as follows:

$$f_D(i) = \begin{cases} i+p' & \text{if } i < q'-1, \\ p'-1 & \text{if } i = q'-1, \\ (i+p'-1) \bmod D & \text{otherwise.} \end{cases}$$

where, $i \in Z_D$. For a permutation f on Z_n, the repeated composition of f with itself is defined inductively as follows: 1) f^0 is the identity permutation; 2) $f^{i+1} = f \circ f^i$. f is said to be cyclic if $f^j(i) \neq i$ for any $i \in Z_n$ and $j \in \{1, 2, \ldots, n-1\}$. The following lemma is proved in [2].

Lemma 1 ([2]): For any positive integers p' and q', $H(d^{p'}, d^{q'}, d)$ is isomorphic to $B(d,D)$ if and only if f_D is cyclic. □

Lemma 2: f_D is cyclic if and only if p' and q' are relatively prime.

Proof of Lemma 2: We define a permutation g_{D+1} on Z_{D+1} as follows:

$$g_{D+1}(i) = (i+p') \bmod (D+1).$$

It is easy to see that

$$g_{D+1}(i) = \begin{cases} D & \text{if } i = q'-1, \\ f_D(q'-1) & \text{if } i = D, \\ f_D(i) & \text{otherwise.} \end{cases}$$

It follows that f_D is cyclic if and only if g_{D+1} is cyclic. By the definition of g_{D+1}, g_{D+1} is cyclic if and only if p' and $D+1 = p'+q'$ are relatively prime. Thus, f_D is cyclic if and only if p' and q' are relatively prime. □

From Lemmas 1 and 2, we obtain the theorem. □

Theorem 2: For any positive integers D and $d \geq 2$, $B(d,D)$ has an OTIS layout. The number of lenses of the OTIS layout is $O(d^{\frac{5}{2}}\sqrt{N})$ if $D \equiv 1 \pmod 4$, $O(d^{\frac{3}{2}}\sqrt{N})$ if $D \equiv 3 \pmod 4$, and $O(d\sqrt{N})$ if D is even, where $N = d^D = |V(B(d,D))|$.

Proof: The theorem follows from Theorem 1 if we choose p' and q' as follows:

$$(p', q') = \begin{cases} (m, m+1) & \text{if D=2m,} \\ (2m'-1, 2m'+1) & \text{if D=4m'-1,} \\ (2m'-1, 2m'+3) & \text{if D=4m'+1,} \\ (1,1) & \text{if D=1.} \end{cases}$$

where, m and m' are positive integers. □

3. Concluding Remarks

- The number of lenses of our OTIS layout is optimal if d is fixed. Closing the gap between the upper and lower bounds on the number of lenses is an open problem.
- It should be noted that not all the digraphs have the OTIS layout. It is an interesting open problem to characterize the digraphs that have OTIS layouts.

References

[1] G.C. Marsden, P.J. Marchand, P. Harvey, and S.C. Esener, "Optical transpose interconnection system architectures," Opt. Lett., vol.18, no.13, pp.1083–1085, 1993.

[2] D. Coudert, A. Ferreira, and S. Perennes, "Isomorphisms of the de Bruijn digraph and free-space optical networks," Netw., vol.40, no.3, pp.155–164, 2002.