On the Two-Dimensional Orthogonal Drawing of Series-Parallel Graphs (Extended Abstract)

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Abstract— **It has been known that every planar 4-graph has a 2-bend 2-D orthogonal drawing with the only exception of octahedron, every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of K_4, and every outerplanar 3-graph with no triangles has a 0-bend 2-D orthogonal drawing. We show in this paper that every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.**

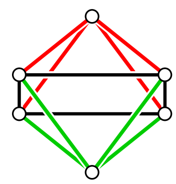
I. Introduction

We consider the problem of generating orthogonal drawings of graphs in the plane. The problem has obvious applications in the design of VLSI circuits and optoelectronic integrated systems: see for example [7], [9].

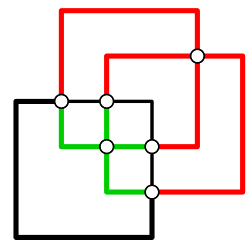
Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of vertices of G. G is called a k-graph if $\Delta(G) \leq k$. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a 2-D drawing of G. A 2-D orthogonal drawing of a planar graph G is a 2-D drawing of G such that each edge is drawn by a sequence of contiguous horizontal and vertical line segments. Notice that a graph G has a 2-D orthogonal drawing only if $\Delta(G) \leq 4$. A 2-D orthogonal drawing with no more than b bends per edge is called a b-bend 2-D orthogonal drawing.

Biedl and Kant [2], and Liu, Morgana, and Simeone [5] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing with only exception of the octahedron shown in Fig. 1(a), which has a 3-bend 2-D orthogonal drawing as shown in Fig. 1(b). Moreover, Kant [4] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of K_4 shown in Fig. 1(c), which has a 2-bend 2-D orthogonal drawing as shown in Fig. 1(d). Nomura, Tayu, and Ueno [6] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia proved that it is $\mathcal{NP}$-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing [3]. Battista, Liotta, and Vargiu showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs [1].

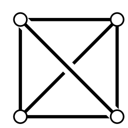
We show in this paper the following theorem.



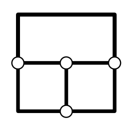
(a) Octahedron.



(b) 3-bend 2-D orthogonal drawing of octahedron.



(c) K_4.



(d) 2-bend 2-D orthogonal drawing of K_4.

Fig. 1. Octahedron, K_4, and their 2-D orthogonal drawings.

Theorem 1: Every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing. □

The proof of Theorem 1 is constructive and provides an $\mathcal{O}(n^2)$ time algorithm to generate such a drawing for an n-vertex series-parallel 4-graph.

II. Preliminaries

A *series-parallel graph* is defined recursively as follows.

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the *terminals*.

(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:

   (i) *Series-composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.

(ii) *Parallel-composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into t. Vertices s and t are the terminals of G.

A series-parallel graph G is naturally associated with a binary tree $T(G)$, which is called a *decomposition tree* of G. The nodes of $T(G)$ are of three types, S-nodes, P-nodes, and Q-nodes. $T(G)$ is defined recursively as follows:

(1) If G is a single edge, then $T(G)$ consists of a single Q-node.

(2-i) If G is obtained from series-parallel graphs G_1 and G_2 by the series-composition, then the root of $T(G)$ is an S-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

(2-ii) If G is obtained from series-parallel graphs G_1 and G_2 by the parallel-composition, then the root of $T(G)$ is a P-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

Notice that the leaves of $T(G)$ are the Q-nodes, and an internal node of $T(G)$ is either an S-node or P-node. Notice also that every P-node has at most one Q-node as a child, since G is a simple graph. If G has n vertices then $T(G)$ has $\mathcal{O}(n)$ nodes, and $T(G)$ can be constructed in $\mathcal{O}(n)$ time [8].

A polygon is said to be rectilinear if every edge of the polygon is parallel to the horizontal axis or the vertical axis. Let Λ and Λ' be rectilinear polygons with distinguished vertices σ and σ', respectively. Λ and Λ' are said to be *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left and right turns for Λ and Λ'.

Let Λ be a rectilinear polygon with distinguished vertices σ and τ, and Λ' be a rectilinear polygon with distinguished vertices σ' and τ'. Λ and Λ' are *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left turns, right turns, and the direction at τ and τ' for Λ and Λ', respectively.

Any two rectilinear rectangles are defined to be *shape-equivalent*.

Let Σ and Σ' be regions bounded by rectilinear polygons Λ and Λ', respectively. Σ and Σ' are said to be *shape-equivalent* if Λ and Λ' are shape-equivalent.

Let Λ_1 and Λ_2 be rectilinear polygons such that Λ_1 is enclosed by Λ_2, and Σ be a region bounded by Λ_1 and Λ_2. Let Λ'_1 and Λ'_2 be rectilinear polygons such that Λ'_1 is enclosed by Λ'_2, and Σ' be a region bounded by Λ'_1 and Λ'_2. Σ and Σ' are *shape-equivalent* if Λ_1 and Λ'_1 are shape-equivalent, and Λ_2 and Λ'_2 are rectilinear rectangles.

A region is said to be rectilinear if it is bounded by rectilinear polygon(s).

III. Proof of Theorem 1 (Sketch)

Let G be a series-parallel 4-graph with terminals s and t. We assume without loss of generality that $d_G(s) \leq d_G(t)$. We generate for G several 1-bend 2-D orthogonal drawings in distinct shape of regions depending on $d_G(s)$ and $d_G(t)$. The number of distinct shapes $\nu(d_G(s), d_G(t))$ is no more than 4 for every pair of $d_G(s)$ and $d_G(t)$.

(a) N_1-drawing. (b) N_2-drawing.

Fig. 3. N-drawings of an edge.

Let Σ be a rectilinear region with distinguished vertices σ and τ. A 1-bend 2-D orthogonal drawing of G in Σ such that s and t are mapped to σ and τ, respectively, is called an *N-drawing* of G generated in Σ. We will show that G has an N-drawing generated in a region shape-equivalent to rectilinear region $\Pi(d_G(s), d_G(t), i)$ shown in Fig. 2, for $1 \leq i \leq \nu(d_G(s), d_G(t))$. It is sufficient to prove the following theorem.

Theorem 2: Every series-parallel 4-graph with terminals s and t has an N-drawing generated in a region shape-equivalent to $\Pi(d_G(s), d_G(t), i)$ for $1 \leq i \leq \nu(d_G(s), d_G(t))$ with the exception that $1 \leq i \leq 2$ if $d_G(s) = d_G(t) = 1$ and $(s,t) \in E(G)$, and $i = 1$ if $d_G(s) = d_G(t) = 3$ and $(s,t) \in E(G)$.

Proof (Sketch): The theorem is proved by induction on $|E(G)|$. An N-drawing of G in a region shape-equivalent to $\Pi(d_G(s), d_G(t), i)$ is called an N_i-drawing of G.

If $|E(G)| = 1$, G is a graph consisting of just an edge (s,t). Such a graph has an N_1-drawing and N_2-drawing as shown in Fig. 3.

Assume that $|E(G)| \geq 2$. We distinguish two cases.

Case 1: G is a series-composition of G_1 and G_2. We can prove the following lemma.

Lemma 1: For any i, $1 \leq i \leq \nu(d_G(s), d_G(t))$, there exist j and k, $1 \leq j \leq \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $1 \leq k \leq \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that an N_i-drawing of G can be generated by combining an N_j-drawing of G_1 and N_k-drawing of G_2. □

Such a pair of j and k for each i is shown in Table I.

Case 2: G is a parallel-composition of G_1 and G_2. We can prove the following lemma.

Lemma 2: For any i, $1 \leq i \leq \nu(d_G(s), d_G(t))$, there exist j and k, $1 \leq j \leq \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $1 \leq k \leq \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that an N_i-drawing of G can be generated by combining an N_j-drawing of G_1 and N_k-drawing of G_2. □

Such a pair of j and k for each i is shown in Table II.

It is tedious but easy to check Tables I and II. The details are omitted here due to space limitation. □

IV. Algorithm (Example)

We can show that the proof of Theorem 2 in the previous section provides an $\mathcal{O}(n^2)$ time recursive algorithm to generate an N-drawing for an n-vertex series-parallel 4-graph. The details are omitted here due to space limitation. As an example, an N_1-drawing of a series-parallel 4-graph G shown in Fig. 4(a) can be generated as follows. G is a parallel-composition of series-parallel graphs G_1 and G_2 shown in

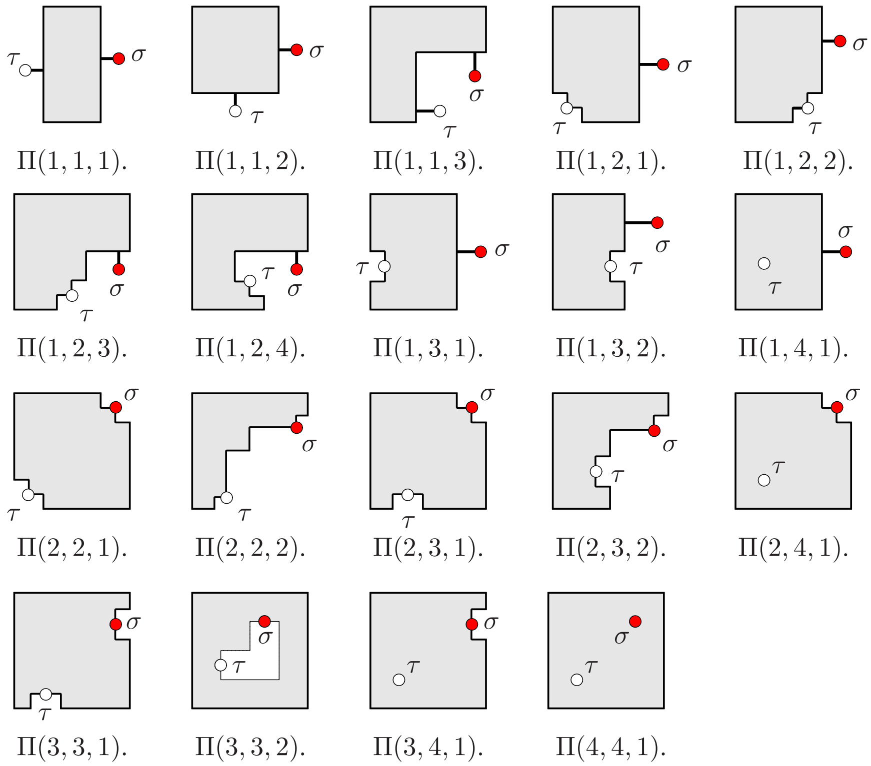


Fig. 2. Rectilinear Regions.

TABLE I

PAIR OF $\Pi(d_{G_1}(s_1), d_{G_1}(t_1), j)$ AND $\Pi(d_{G_2}(s_2), d_{G_2}(t_2), k)$ FOR $\Pi(d_G(s), d_G(t), i)$ WHEN G IS A SERIES-COMPOSITION, WHERE $1 \leq \gamma \leq 4$.

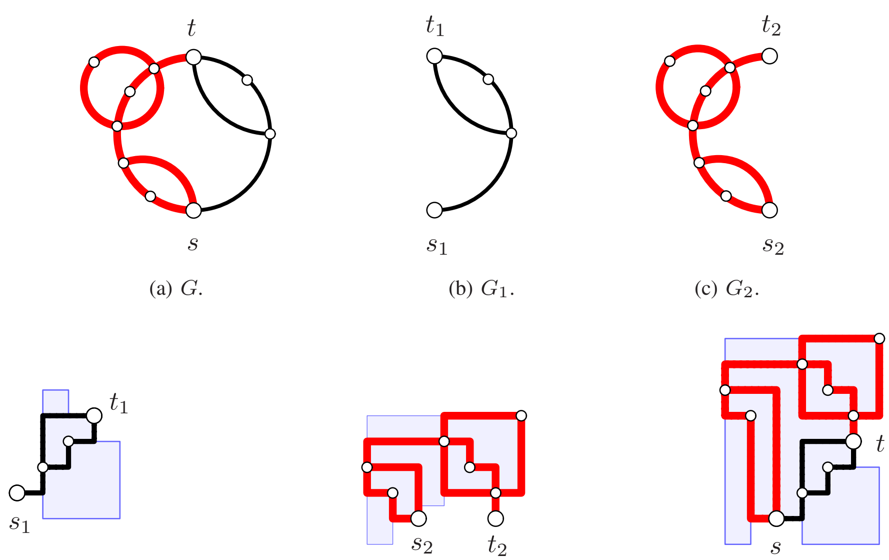
$\Pi(d_G(s), d_G(t), i)$	$\Pi(d_{G_1}(s_1), d_{G_1}(t_1), j)$	$\Pi(d_{G_2}(s_2), d_{G_2}(t_2), k)$
$\Pi(1,1,1)$	$\Pi(1, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 1, 1)$
$\Pi(1,1,2)$	$\Pi(1,1,2)$	$\Pi(d_{G_2}(s_2), 1, 1)$
	$\Pi(1,2,1)$	$\Pi(d_{G_2}(s_2), 1, 2)$
	$\Pi(1,3,1)$	$\Pi(1,1,2)$
$\Pi(1,1,3)$	$\Pi(1, d_{G_1}(t_1), 2)$	$\Pi(d_{G_2}(s_2), 1, 2)$
$\Pi(1,2,1)$	$\Pi(1, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 2, 1)$
$\Pi(1,2,2)$	$\Pi(1, d_{G_1}(t_1), 2)$	$\Pi(d_{G_2}(s_2), 2, 1)$
$\Pi(1,2,3)$	$\Pi(1,1,1)$	$\Pi(d_{G_2}(s_2), 2, 2)$
	$\Pi(1,2,3)$	$\Pi(d_{G_2}(s_2), 2, 1)$
	$\Pi(1,3,2)$	$\Pi(1,2,1)$
$\Pi(1,2,4)$	$\Pi(1, d_{G_1}(t_1), 2)$	$\Pi(d_{G_2}(s_2), 2, 2)$
$\Pi(1,3,1)$	$\Pi(1,1,2)$	$\Pi(d_{G_2}(s_2), 3, 1)$
	$\Pi(1,2,1)$	$\Pi(d_{G_2}(s_2), 3, 1)$
	$\Pi(1,3,1)$	$\Pi(1,3,1)$
$\Pi(1,3,2)$	$\Pi(1, d_{G_1}(t_1), 2)$	$\Pi(d_{G_2}(s_2), 3, 1)$
$\Pi(2,2,1)$	$\Pi(2, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 2, 1)$
$\Pi(2,2,2)$	$\Pi(2, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 2, 2)$
$\Pi(2,3,1)$	$\Pi(2, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 3, 1)$
$\Pi(2,3,2)$	$\Pi(2, d_{G_1}(t_1), 2)$	$\Pi(d_{G_2}(s_2), 3, 1)$
$\Pi(3,3,1)$	$\Pi(3, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 3, 1)$
$\Pi(3,3,2)$	$\Pi(3,1,2)$	$\Pi(d_{G_2}(s_2), 3, 1)$
	$\Pi(3,2,2)$	$\Pi(d_{G_2}(s_2), 3, 1)$
	$\Pi(3,3,1)$	$\Pi(1,3,2)$
$\Pi(\gamma,4,1)$	$\Pi(\gamma, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), 4, 1)$
$\Pi(4,\gamma,1)$	$\Pi(4, d_{G_1}(t_1), 1)$	$\Pi(d_{G_2}(s_2), \gamma, 1)$

Fig. 4(b) and Fig. 4(c), respectively. Since $d_G(s) = d_G(t) = 3$, we need an N_1-drawing Γ_1 of G_1 and N_3-drawing Γ_2 of G_2 by Table II. Γ_1 and Γ_2 are shown in Fig. 4(d) and Fig. 4(e), respectively. Finally, an N_1-drawing of G can be generated by scaling Γ_1 and Γ_2 appropriately and identifying s_1 with s_2, and t_1 with t_2 as shown in Fig. 4(f).

TABLE II

PAIR OF $\Pi(d_{G_1}(s_1), d_{G_1}(t_1), j)$ AND $\Pi(d_{G_2}(s_2), d_{G_2}(t_2), k)$ FOR $\Pi(d_G(s), d_G(t), i)$ WHEN G IS A PARALLEL-COMPOSITION.

$\Pi(d_G(s), d_G(t), i)$	$\Pi(d_{G_1}(s_1), d_{G_1}(t_1), j)$	$\Pi(d_{G_2}(s_2), d_{G_2}(t_2), k)$
$\Pi(2, 2, 1)$	$\Pi(1, 1, 2)$	$\Pi(1, 1, 2)$
$\Pi(2, 2, 2)$	$\Pi(1, 1, 2)$	$\Pi(1, 1, 3)$
$\Pi(2, 3, 1)$	$\Pi(1, 1, 2)$	$\Pi(1, 2, 2)$
$\Pi(2, 3, 2)$	$\Pi(1, 1, 2)$	$\Pi(1, 2, 4)$
$\Pi(2, 4, 1)$	$\Pi(1, 1, 2)$	$\Pi(1, 3, 2)$
$\Pi(2, 4, 1)$	$\Pi(1, 2, 2)$	$\Pi(1, 2, 2)$
$\Pi(3, 3, 1)$	$\Pi(1, 1, 2)$	$\Pi(2, 2, 2)$
$\Pi(3, 3, 1)$	$\Pi(1, 2, 1)$	$\Pi(2, 1, 3)$
$\Pi(3, 3, 2)$	$\Pi(1, 1, 3)$	$\Pi(2, 2, 2)$
$\Pi(3, 3, 2)$	$\Pi(1, 2, 2)$	$\Pi(2, 1, 4)$
$\Pi(3, 4, 1)$	$\Pi(1, 1, 2)$	$\Pi(2, 3, 2)$
$\Pi(3, 4, 1)$	$\Pi(1, 2, 2)$	$\Pi(2, 2, 2)$
$\Pi(3, 4, 1)$	$\Pi(1, 3, 2)$	$\Pi(2, 1, 2)$
$\Pi(4, 4, 1)$	$\Pi(1, 1, 2)$	$\Pi(3, 3, 2)$
$\Pi(4, 4, 1)$	$\Pi(1, 2, 1)$	$\Pi(3, 2, 2)$
$\Pi(4, 4, 1)$	$\Pi(1, 3, 1)$	$\Pi(3, 1, 1)$
$\Pi(4, 4, 1)$	$\Pi(2, 2, 2)$	$\Pi(2, 2, 2)$



(a) G. (b) G_1. (c) G_2.

(d) N_1-drawing Γ_1 of G_1. (e) N_3-drawing Γ_2 of G_2. (f) N_1-drawing of G.

Fig. 4. Example of a Recursive Step of Algorithm.

We conclude with some remarks. We learned recently that Zhou and Nishizeki proposed a linear time algorithm to generate a 1-bend 2-D orthogonal drawing for a series-parallel 3-graph[10]. We can prove that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing, which will appear in a forthcoming paper.

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