On the Three-Dimensional Layout of Hypercubes

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1 Introduction

The hypercube has been known as one of the most important network architectures for parallel computing, and extensively studied in the literature. This paper shows an efficient 3-D layout of the hypercube.

The n-*cube* (*n-dimensional cube*) $Q(n)$ is the graph with $N = 2^n$ vertices labeled 0 through $N-1$ such that two vertices are jointed by an edge if and only if their labels in the binary representation differ by exactly one bit. $\mathcal{R}(m_x, m_y, m_z)$ is a 3-D *grid* with m_x, m_y and m_z vertices along x, y and z dimensions, respectively. $\mathcal{R}(m_x, m_y, 2)$ is also called a 2-D grid.

An *embedding* $\langle \phi, \rho \rangle$ of a graph $\mathcal{G}$ into a graph $\mathcal{H}$ consists of a one-to-one mapping $\phi : V(\mathcal{G}) \to V(\mathcal{H})$, together with a mapping ρ that maps each edge $(u, v) \in E(\mathcal{G})$ onto a path $\rho(u, v)$ in $\mathcal{H}$ that connects vertices $\phi(u)$ and $\phi(v)$. An embedding $\langle \phi, \rho \rangle$ of a graph $\mathcal{G}$ into a 3-D grid $\mathcal{R}$ is called a 3-D *layout* of $\mathcal{G}$ if routing paths $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint for any distinct $e_1, e_2 \in E(\mathcal{G})$. $|V(\mathcal{R})|$ is called the *volume* of the 3-D layout, and $\mathrm{vol}(\mathcal{G})$ is the minimum volume of a 3-D layout of $\mathcal{G}$. If $\mathcal{R}$ is a 2-D grid then the layout is also called a 2-D layout.

The *bisection width* $\mathrm{bw}(\mathcal{G})$ of a graph $\mathcal{G}$ is the minimum number of edges that must be removed from $\mathcal{G}$ in order to partition $\mathcal{G}$ into two equal-sized subgraphs to within one vertex. It is well-known that $\mathrm{vol}(\mathcal{G}) \geq \mathrm{bw}(\mathcal{G})^{3/2}$ for any graph $\mathcal{G}$ [5]. Since it is also well-known that $\mathrm{bw}(Q(n)) = \Theta(N)$ [2],

$$\mathrm{vol}(Q(n)) = \Omega(N^{3/2}).$$

This paper shows that

$$\mathrm{vol}(Q(n)) = O(N^{3/2}),$$

which is obtained by an explicit 3-D layout of $Q(n)$ based on an embedding of $Q(n)$ into a 3-D grid introduced in [6], and an efficient 2-D linear layout considered in [1], [3] and [4]. This is the first explicit 3-D layout of $Q(n)$ with optimal volume, as far as the authors know.

2 2-D Linear Laoyut of $Q(n)$

In the 2-D layout of $Q(n)$, each vertex is represented by a 2-D grid $\mathcal{R}(l, l, 2)$, where $l \geq n = \log N$. In the 2-D linear layout of $Q(n)$, all 2-D grids representing vertices of $Q(n)$ are laid out in a linear array, side by side. The following theorem was proved in [1], [3] and [4].

Theorem I *$Q(n)$ can be linearly laid out in a 2-D grid $\mathcal{R}(l \times N, l + N, 2)$.* ∎

Such a layout of $Q(3)$ is shown in Fig. 1.

3 3-D Linear Layout of $Q(n)$

In the 3-D linear layout of $Q(n)$, each vertex is represented by a 3-D grid $\mathcal{R}(\lceil N^{1/2} \rceil, \lceil N^{1/2} \rceil, \lceil N^{1/2} \rceil)$, and all such grids are laid out in a linear array, side by side. We can prove the following theorem by using Theorem I.

Theorem 1 *$Q(n)$ can be linearly laid out in a 3-D grid $\mathcal{R}(\lceil N^{1/2} \rceil \times N, \lceil N^{1/2} \rceil + \lceil N^{1/2} \rceil, \lceil N^{1/2} \rceil)$.* ∎

Such a layout of $Q(3)$ is shown in Fig. 2.

4 3-D Layout of $Q(n)$

In the 3-D layout of $Q(n)$, each vertex is represented by a 3-D grid $\mathcal{R}(\lceil N^{1/6} \rceil, \lceil N^{1/6} \rceil, \lceil N^{1/6} \rceil)$. The binary representation of a vertex of $Q(n)$ is trisected to obtain its coordinates. Let $a = \lfloor (n + 1)/3 \rfloor$, and $b = n - 2a$. The value of the least significant a bits represents the x-coordinate, the value of the next a bits represents the y-coordinate, and the value of the most significant b bits represents the z-coordinate. Using the coordinates, we arrange the 3-D grids representing the vertices of $Q(n)$ in a 2^a by 2^a by 2^b array with $\lceil N^{1/6} \rceil$ spacing in between. Such an arrangement of vertices for $Q(5)$ is illustrated in Fig. 3.

The vertices with the same y- and z-coordinate values induce an a-cube, which is laid out in a 3-D grid $\mathcal{R}((\lceil A^{1/2}\rceil + \lceil N^{1/6}\rceil) \times A, \lceil A^{1/2}\rceil + \lceil A^{1/2}\rceil, \lceil A^{1/2}\rceil)$, using the 3-D linear layout in the previous section, where $A = 2^a$. Since $A = O(N^{1/3})$, we couclude that the a-cube can be laid out in a 3-D grid $\mathcal{R}(O(N^{1/2}), O(N^{1/6}), O(N^{1/6}))$. Similarly, an a-cube induced by the vertices with the same z- and x-coordinate values can be laid out in a 3-D grid $\mathcal{R}(O(N^{1/6})$, $O(N^{1/2})$, $O(N^{1/6}))$ and a b-cube induced by the vertices with the same x- and y-coordinate values can be laid out in a 3-D grid $\mathcal{R}(O(N^{1/6})$, $O(N^{1/6})$, $O(N^{1/2}))$.

By combining all such 3-D linear layouts of subcubes, $Q(n)$ can be laid out in a 3-D grid $\mathcal{R}(O(N^{1/2})$, $O(N^{1/2})$, $O(N^{1/2}))$. Fig. 4 illustrates such a 3-D layout of $Q(5)$, where shaded cubes represent vertices, and the remaining region is used for wiring. Thus we obtain the following theorem.

Theorem 2 $\mathrm{vol}(Q(n)) = O(N^{3/2})$. ∎

References

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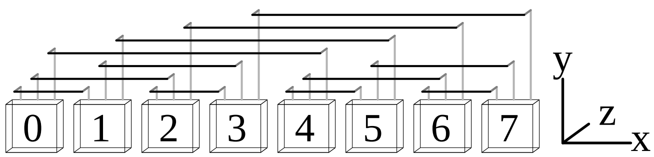

Fig. 1. 2-D linear layout of $Q(3)$.

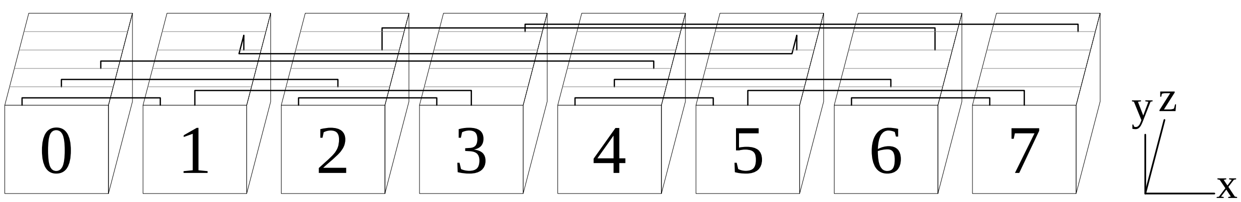

Fig. 2. 3-D linear layout of $Q(3)$.

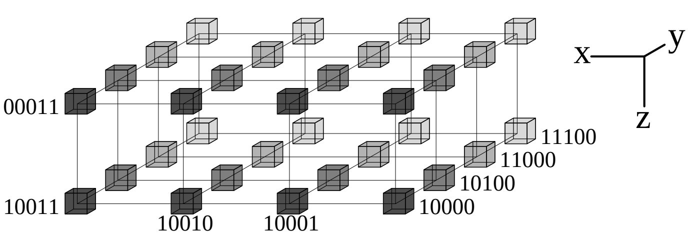

Fig. 3. Arrangement of vertices for $Q(5)$.

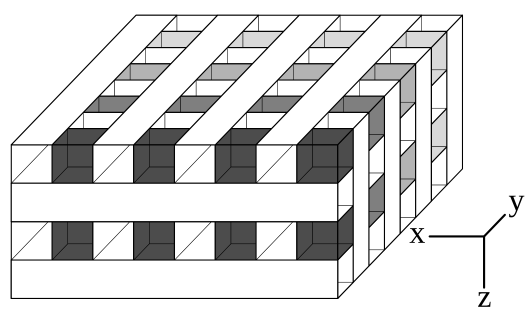

Fig. 4. 3-D layout of $Q(5)$.