Three-Dimensional Channel Routing is in $\mathcal{NP}$

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1 Introduction

It is mentioned in our previous paper [4] that the 3-D channel routing is $\mathcal{NP}$-complete, and the proof of the $\mathcal{NP}$-hardness is outlined there. The purpose of this paper is to show that the 3-D channel routing is in $\mathcal{NP}$, complementing the previous paper.

The 3-D channel is a 3-D grid G consisting of *columns*, *rows*, and *layers* which are rectilinear grid planes defined by fixing x-, y-, and z-coordinates at integers, respectively. The numbers of columns, rows, and layers are called the *width*, *depth*, and *height* of G, respectively. (See Fig. 1.) G is called a (W, D, H)-*channel* if the width is W, depth is D, and height is H. A vertex of G is a grid point with integer coordinates. We assume without loss of generality that the vertex set of a (W, D, H)-channel is $\{(x, y, z)|x \in [W], y \in [D], z \in [H]\}$, where $[n] = \{1, 2, \dots, n\}$ for a positive integer n. Layers defined by $z = H$ and $z = 1$ are called the *top* and *bottom layers*, respectively.

A *terminal* is a vertex of G located in the top or bottom layer. A *net* is a set of terminals to be connected. A net containing k terminals is called a k-*net*. The object of the 3-D channel routing problem is to connect the terminals in each net with a tree in G using as few layers as possible and as short wires as possible in such a way that trees spanning distinct nets are vertex-disjoint. A set of nets is said to be *routable* in G if G has vertex-disjoint trees spanning the nets. A set of nets of a (W, D, H)-channel is said to be *routable* with height H if it is routable in the (W, D, H)-channel.

We consider the following decision problem.

3-D CHANNEL ROUTING

INSTANCE: Positive integers W, D, H, p, q, a set of terminals $T \subseteq \{(a_i, a'_i, H)|a_i \in [W], a'_i \in [D], i \in [p]\} \cup \{(b_j, b'_j, 0)|b_j \in [W], b'_j \in [D], j \in [q]\}$, and a partition of T into nets $N_1, N_2, \dots, N_\nu$.

QUESTION: Is the set of nets $\mathcal{N} = \{N_1, N_2, \dots, N_\nu\}$ routable in a (W, D, H)-channel G?

The purpose of this paper is to show the following.

Theorem 1 3-D CHANNEL ROUTING *is in* $\mathcal{NP}$. ∎

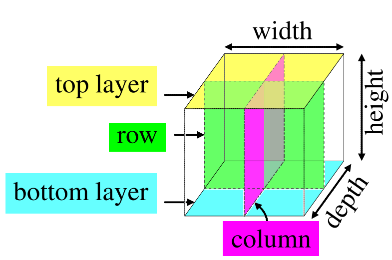


Figure 1: The three-dimensional channel.

2 Proof Outline of Theorem 1

Suppose that $\mathcal{N}$ is routable in G. Let $\tau_i = |N_i|$, and $\tau = \sum_{i=1}^{\nu} \tau_i$. Notice that $\tau = |T| = p + q$. A tree connecting the terminals in each net N_i is a rectilinear Steiner tree S_i spanning the terminals in N_i. S_i can be represented by the coordinates of terminals, Steiner points, and bends. Notice that the number of Steiner points of S_i is at most $\tau_i - 1$. Thus, in order to prove that 3-D CHANNEL ROUTING is in $\mathcal{NP}$, it suffices to show that $\mathcal{N}$ is routable in G with polynomially bounded number of bends.

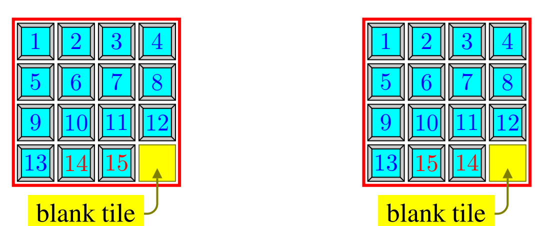
We first show that if $\mathcal{N}$ is routable with a finite height, $\mathcal{N}$ is also routable with a polynomially bounded height.

2.1 Routability with Polynomial Height

2.1.1 The Case of 2-Nets

We first consider the problem for 2-nets such that a terminal of each net is on the top layer and the other on the bottom layer. The 3-D channel routing for 2-nets is closely related to the $(r \times s)$-puzzle defined below.

The $(r \times s)$-puzzle is a generalization of the well-known 15-puzzle [1]. The $(r \times s)$-*puzzle* is played on an $r \times s$ board. There are rs distinct tiles on the board: one *blank tile* and $rs - 1$ tiles numbered from 1 to $rs - 1$. Each of the rs square locations of the board is occupied by exactly one tile. An instance of $(r \times s)$-puzzle consists of two board configurations B_1 (the *initial configuration*) and B_2 (the *final configuration*). A *move* is an exchange of the blank tile with a tile located on a horizontally or vertically adjacent location. The goal of the puzzle is to find a sequence of moves that transforms B_1 to B_2. The configuration B_2 is said to be *reachable* from B_1 if there exists such a sequence of moves. Notice that B_2 is reachable from B_1 if and only if B_1 is reachable from B_2. Figure 2 shows two unreachable configurations of (4×4)-puzzle. This is the original 15-puzzle of Loyd [1]. Our problem is to find a shortest sequence of moves that transforms B_1 to B_2 if B_1 and B_2 are reachable. The corresponding decision problem is described as follows.



(a) Initial configuration (b) Final configuration

Figure 2: Unreachable configurations of (4×4)-puzzle.

$(r \times s)$-PUZZLE

INSTANCE: Two $r \times s$ board configurations B_1 and B_2, and a positive integer k.

QUESTION: Is there a sequence of at most k moves that transforms B_1 to B_2?

The configurations B_1 and B_2 are said to be *reachable* with k moves if there exists a sequence of at most k moves that transforms B_1 to B_2. It is easy to see the following.

Lemma 1 *If B_1 and B_2 are reachable then they are reachable with $\mathcal{O}(r^2 s + rs^2)$ moves.* ∎

The $(r \times s)$-puzzle can be considered as a 3-D channel routing for 2-nets. The configurations B_1 and B_2 are corresponding to the top and bottom layers. A nonblank tile is corresponding to a 2-net. It is easy to see the following.

Theorem 2 *B_1 and B_2 are reachable with k moves if and only if the 2-nets corresponding to the nonblank tiles are routable in an (r, s, k)-channel.* ∎

It should be noted that the total number of bends of the routing for 2-nets in Theorem 2 is $\mathcal{O}(k)$. Thus by Lemma 1 and Theorem 2, we obtain the following.

Lemma 2 *If $\nu = WD - 1$, and $\mathcal{N}$ is consisting of 2-nets and routable in G then $\mathcal{N}$ is routable with $\mathcal{O}(\nu^2)$ height and $\mathcal{O}(\nu^2)$ bends.* ∎

We can easily extend Lemma 2 as follows.

Lemma 3 *If $\nu = \Omega(WD)$, and $\mathcal{N}$ is consisting of 2-nets and routable in G then $\mathcal{N}$ is routable with $\mathcal{O}(\nu^2)$ height and $\mathcal{O}(\nu^2)$ bends.* ∎

Moreover, we can prove the following by using an easy extension of a 3-D channel routing algorithm presented in [3].

Lemma 4 *If $\nu \leq \lfloor W/2 \rfloor \lfloor D/2 \rfloor$, and $\mathcal{N}$ is consisting of 2-nets and routable in G then $\mathcal{N}$ is routable with $\mathcal{O}(\nu^2)$ height and $\mathcal{O}(\nu^2)$ bends.* ∎

From Lemmas 3 and 4, we obtain the following.

Theorem 3 *If $\mathcal{N}$ is consisting of ν 2-nets and routable in G then $\mathcal{N}$ is routable with $\mathcal{O}(\nu^2)$ height and $\mathcal{O}(\nu^2)$ bends.* ∎

2.1.2 The General Case

Let N_i^t and N_i^b be the terminals of N_i on the top layer and bottom layer, respectively. Let $\mathcal{N}^t = \{N_1^t, N_2^t, \ldots, N_\nu^t\}$ and $\mathcal{N}^b = \{N_1^b, N_2^b, \ldots, N_\nu^b\}$. We can prove that $\mathcal{N}^t$ is routable with $\mathcal{O}(p^2)$ height and $\mathcal{O}(p^2)$ bends, and $\mathcal{N}^b$ is routable with $\mathcal{O}(q^2)$ height and $\mathcal{O}(q^2)$ bends. It follows that the general case can be reduced to the case of 2-nets. Since p and q are $\mathcal{O}(\tau)$, we conclude from Theorem 3 that if $\mathcal{N}$ is routable in G then $\mathcal{N}$ is routable with $\mathcal{O}(\tau^2)$ height and $\mathcal{O}(\tau^2)$ bends. Thus, we obtain the following.

Theorem 4 *If $\mathcal{N}$ is routable in G then $\mathcal{N}$ is routable with height $f(\tau)$ and $b(\tau)$ bends, where $f(\tau)$ and $b(\tau)$ are some functions of $\mathcal{O}(\tau^2)$.* ∎

Now, we are ready to prove Theorem 1.

2.2 Routability with Polynomial Number of Bends

Suppose $\mathcal{N}$ is routable in G. If $H \geq f(\tau)$ then $\mathcal{N}$ is routable in G with $\mathcal{O}(\tau^2)$ bends by Theorem 4.

Assume $H < f(\tau)$. Let $\eta_1, \eta_2, \ldots, \eta_\lambda$ and $\psi_1, \psi_2, \ldots, \psi_{\lambda'}$ be the increasing sequences of x-coordinates and y-coordinates of terminals, respectively. By definition, if a terminal is located at (x, y, z), $x = \eta_i$ and $y = \psi_j$ for some $i \in [\lambda]$ and $j \in [\lambda']$. Let $\eta_0 = \psi_0 = 1$, $\eta_{\lambda+1} = W$, and $\psi_{\lambda'+1} = D$. Since $\mathcal{N}$ is routable in G, there exist vertex-disjoint Steiner trees S_k for N_k, $k \in [\nu]$. Let $\mathcal{S} = \{S_1, S_2, \ldots, S_\nu\}$.

For each $m \in [\lambda + 1]$, let G_m^X be a subgrid of G induced by the vertices in $\{(x, y, z) | \eta_{m-1} \leq x \leq \eta_m, y \in [D], z \in [H]\}$. If $\eta_m - \eta_{m-1} \geq f(\tau)$, we can reroute $\mathcal{S}$ in G_m^X so that the number of bends in G_m^X is $\mathcal{O}(\tau^2)$ by using Theorem 4. Here, the columns defined by $x = \eta_{m-1}$ and $x = \eta_m$ are considered as the top and bottom layers of a 3-D channel.

Similarly, for each $m' \in [\lambda' + 1]$, let $G_{m'}^Y$ be a subgrid of G induced by the vertices in $\{(x, y, z) | x \in [W], \psi_{m'-1} \leq y \leq \psi_{m'}, z \in [H]\}$. If $\psi_{m'} - \psi_{m'-1} \geq f(\tau)$, we can reroute $\mathcal{S}$ in $G_{m'}^Y$ so that the number of bends in $G_{m'}^Y$ is $\mathcal{O}(\tau^2)$.

Let $\mathcal{X}$ be the union of G_m^X with $\eta_m - \eta_{m-1} \geq f(\tau)$, $\mathcal{Y}$ be the union of $G_{m'}^Y$ with $\psi_{m'} - \psi_{m'-1} \geq f(\tau)$, $\overline{\mathcal{X}} = G \setminus \mathcal{X}$, and $\overline{\mathcal{Y}} = G \setminus \mathcal{Y}$. After the rerouting, the total number of bends in $\mathcal{X} \cup \mathcal{Y}$ is $\mathcal{O}(\lambda\tau^2 + \lambda'\tau^2) = \mathcal{O}(\tau^3)$. Moreover, the number of bends in $\overline{\mathcal{X}} \cap \overline{\mathcal{Y}}$ is at most $|V(\overline{\mathcal{X}} \cap \overline{\mathcal{Y}})| = \mathcal{O}(\lambda\lambda' f(\tau)^2 H) = \mathcal{O}(\tau^8)$. Thus, we conclude that $\mathcal{N}$ is routable with $\mathcal{O}(\tau^8)$ bends.

This completes the proof of Theorem 1.

3 Concluding Remarks

(1) Lemma 1 for a special case of $r = s$ is mentioned in [2].

(2) We learned recently that $(n \times n)$-PUZZLE is $\mathcal{NP}$-complete [2] . The result and Theorems 1 and 2 indicate that 3-D CHANNEL ROUTING is $\mathcal{NP}$-complete even for 2-nets.

References

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