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On the Quantum Query Complexity of All-Pairs Shortest Paths

Fuwei Cai, Satoshi Tayu, and Shuichi Ueno
Department of Communications and Integrated Systems, Tokyo Institute of Technology

1 Introduction

We show lower bounds for the quantum query complexity of the all-pairs shortest paths problem (APSP) for non-negatively weighted directed graphs (digraphs), both in the adjacency matrix model and in an adjacency list-like array model.

For the single-source shortest paths problem (SSSP) for a weighted digraph with n vertices and m directed edges (arcs), the classical complexity of $\Theta(m+n\log n)$ has been known in the literature. The upper bound is derived from an implementation of Dijkstra's algorithm [1] with Fibonacci heaps, and the lower bound is shown by Pettie [5]. It has been also known that the classical complexity of APSP is $O(mn + n^2 \log\log n)$ [5] and $\Omega(mn)$ [4].

For SSSP for a weighted digraph with n vertices and m arcs, the quantum query complexity is $O(n^{3/2}\log^2 n)$ and $\Omega(n^{3/2})$ in the matrix model, and $O(\sqrt{mn}\log^2 n)$ and $\Omega(\sqrt{mn})$ in the array model [2].

From the bounds for SSSP, we have trivial bounds for the query complexity of APSP as follows: $O(n^{5/2}\log^2 n)$ and $\Omega(n^{3/2})$ in the matrix model, and $O(\sqrt{mn^3}\log^2 n)$ and $\Omega(\sqrt{mn})$ in the array model. Furrow improved the upper bounds to $O(n^{5/2}\log n)$ in the matrix model and $O(\sqrt{mn^3}\log n + n^2\log^3 n)$ in the array model [3].

This paper shows non-trivial lower bounds for the quantum query complexity of APSP by proving the following theorem.

Theorem 1 APSP requires $\Omega(n^2)$ queries in the matrix model, and $\Omega(\sqrt{m^3/n^2})$ queries in the array model.

It should be noted that our lower bound in the array model is better than the trivial one if $m = \omega(n^{3/2})$. It is an interesting open question to close the gap between the upper and lower bounds.

2 Preliminaries

The query complexity of a graph problem is the minimum number of queries to the graph required for solving the problem. We consider two query models for a non-negatively weighted digraph G with vertices $\{v_0, v_1, \ldots, v_{n-1}\}$. In the matrix model, G is given as the weight matrix W, where $W_{i,j}$ is the weight of arc (v_i, v_j) if exists, and ∞ if (v_i, v_j) is not an arc of G. In the array model, G is given by a sequence of functions $f_i : [\deg^+_G(v_i)] \longrightarrow [n] \times \mathbb{R}^+$, $i \in [n]$, such that if $f_i(j) = (k, w)$, then there is an arc (v_i, v_k) with weight w, where $\deg^+_G(v_i)$ is the out-degree of v_i, $[n]$ denotes the set $\{0, 1, \ldots, n-1\}$, and $\mathbb{R}^+$ is the set of non-negative real numbers.

We use the following theorem shown in [2].

Theorem I: The problem of finding the minimum entry in every row of an $r \times c$ matrix with non-negative entries requires $\Omega(r\sqrt{c})$ queries.

3 Proof of Theorem 1

Let M be a $d^2 \times c$ matrix with non-negative entries. From Theorem I, $\Omega(d^2\sqrt{c})$ queries are required to find the minimum entry in every row of M.

We construct a weighted digraph G with n vertices and m arcs from M as follows. The vertex set of G is

$$\{u_i | i \in [d]\} \cup \{u_{i,j} | i \in [d], j \in [c]\} \cup \{v_{i,j} | i \in [d], j \in [c]\} \cup \{v_i | i \in [d]\},$$

the arc set of G is

$$\{(u_i, u_{i,j}) | i \in [d], j \in [c]\} \cup \{(u_{i,j}, v_{k,j}) | i, k \in [d], j \in [c]\} \cup \{(v_{i,j}, v_i) | i \in [d], j \in [c]\},$$

the weight of arc $(u_{i,j}, v_{k,j})$ is

$$M_{di+k,j} \quad (i, k \in [d], j \in [c]),$$

and the weight of any other arc is 0.

It is easy to see that the weight of the shortest (u_i, v_k)-path is corresponding to the minimum entry of the $(di + k)$-th row of M, $i, k \in [d]$. Since

$$n = d + cd + cd + d = \Theta(cd), \text{ and}$$
$$m = cd + cd^2 + cd = \Theta(cd^2),$$

we conclude that APSP requires $\Omega(\sqrt{m^3/n^2})$ queries in the array model, and so $\Omega(n^2)$ queries in the matrix model.

References

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