Orthogonal Ray Graphs and Nano-PLA Design

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***Abstract*— The logic mapping problem and the problem of finding a largest square sub-crossbar with no defects in a nano-crossbar with nonprogrammable crosspoint defects and disconnected wire defects have been known to be NP-hard. This paper shows that for nano-crossbars with only disconnected wire defects, the former remains NP-hard, while the latter can be solved in polynomial time.**

I. INTRODUCTION

The problem of mapping a logic function onto a defective nano-crossbar with nonprogrammable crosspoint defects and disconnected wire defects was first considered by Rao, Orailoglu, and Karri [4]. They proposed several heuristics since the problem is NP-hard. The problem of finding a maximum defect-free square sub-crossbar in a nano-crossbar with nonprogrammable crosspoint defects and disconnected wire defects was first investigated by Tahoori [6]. Since the problem is also NP-hard, several heuristics have been proposed [1], [6].

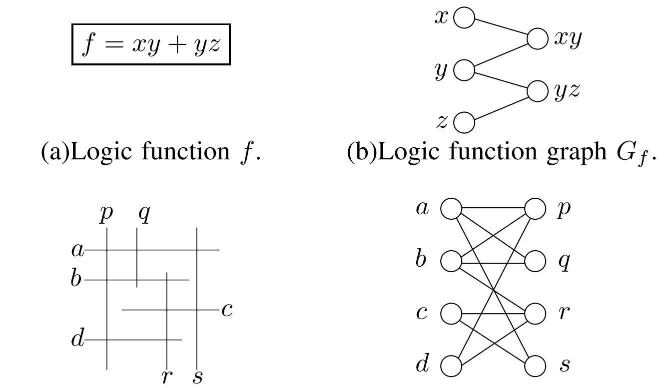
This paper considers the complexity of the problems for nano-crossbars with only disconnected wire defects.

I-A. LOGIC MAPPING

Let f be a logic function in a sum-of-product form. The problem of implementing f in a surviving sub-crossbar S of a nano-crossbar with disconnected wire defects is formulated as LOGIC MAPPING, which is the problem of assigning the literals and product terms of f to surviving nano-wires of S so that containment relationships among the literals and product terms can be represented by crosspoint connections in S. A graph model of LOGIC MAPPING can be obtained as follows.

Let L_f be the set of literals of f, and P_f be the set of product terms of f. A logic function graph G_f for f is a bipartite graph defined as follows: $V(G_f) = L_f \cup P_f$, and (L_f, P_f) is a bipartition of G_f; vertices $l \in L_f$ and $p \in P_f$ are connected by an edge if and only if literal l is contained in product term p.

Let W_h be the set of surviving horizontal nano-wires, and W_v be the set of surviving vertical nano-wires of S. A surviving sub-crossbar graph G_S for S is a bipartite graph defined as follows: $V(G_S) = W_h \cup W_v$ and (W_h, W_v) is a bipartition of G_S; vertices $x \in W_h$ and $y \in W_v$ are connected by an edge if and only nano-wires x and y have a crosspoint. Then, LOGIC MAPPING can be modeled as the subgraph isomorphism problem, which is to find a subgraph of G_S isomorphic to G_f. An example of a logic function f, a defective crossbar S, and their corresponding bipartite graphs G_f and G_S is shown in Figure 1.



(a)Logic function f. (b)Logic function graph G_f.

(c) Surviving crossbar S. (d) Surviving crossbar graph G_S.

Fig. 1. An instance of LOGIC MAPPING and the corresponding graphs.

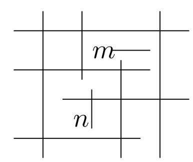


Fig. 2. Nano-wires such as m and n are unusable.

I-B. SQUARE SUB-CROSSBAR

SQUARE SUB-CROSSBAR is the problem of finding a maximum defect-free square sub-crossbar within the original nano-crossbar with disconnected wire defects. SQUARE SUB-CROSSBAR can be modeled as the balanced complete bipartite subgraph problem, which is to find a complete bipartite graph $K_{k,k}$ contained in G_S.

I-C. OUR RESULTS

Although it is well known that both the subgraph isomorphism problem and the balanced complete bipartite subgraph problem are NP-hard for bipartite graphs [2], [3], the complexity of LOGIC MAPPING and SQUARE SUB-CROSSBAR is not clear since the graphs representing surviving sub-crossbars are a special kind of bipartite graphs.

A bipartite graph G with a bipartition (U, V) is called an *orthogonal ray graph* if there exist a family of non-intersecting rays (half-lines) $R_u, u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting rays $R_v, v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect.

Nano-wires such as m and n of a defective nano-crossbar shown in Figure 2 cannot be controlled as they do not touch the

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boundary of the originally intended nano-crossbar. Since we cannot use such nano-wires, a graph representing a surviving sub-crossbar must be an orthogonal ray graph. The orthogonal ray graph was introduced by Shrestha, Kobayashi, Tayu, and Ueno [5] as a graph model for a surviving sub-crossbar.

We show in Section III that LOGIC MAPPING is NP-hard by showing that the subgraph isomorphism problem is NP-hard even for orthogonal ray graphs. We also show in Section IV that SQUARE SUB-CROSSBAR can be solved in polynomial time provided that the vertices of the orthogonal ray graph representing a surviving sub-crossbar are ordered so as to reflect the position of nano-wires relative to each other, which is a quite natural condition.

II. Orthogonal Ray Graphs

Let G be an orthogonal ray graph with a bipartition (U, V). G is called a *two-directional orthogonal ray graph* if $R_u = \{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v = \{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where a_w and b_w are real numbers for any $w \in U \cup V$. The 3-claw is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. (See Figure 3(a).)

Although the following characterization of two-directional orthogonal ray trees was shown in [5], we show complete proofs to make the paper self-contained.

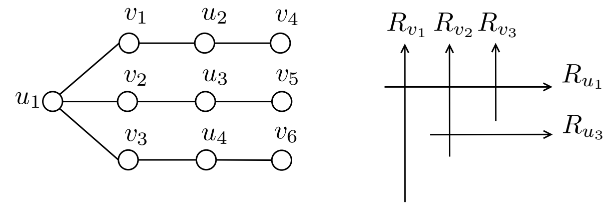
Lemma 1: The 3-claw is not a 2-directional orthogonal ray graph.

Proof: Assume to the contrary that the 3-claw is a 2-directional orthogonal ray graph. Let the vertices of the *3-claw* be named as in Figure 3(a). We shall refer to the endpoint of the ray corresponding to a vertex v as (a_v, b_v). Without loss of generality, suppose R_{u_1} is a horizontal ray and that R_{v_1}, R_{v_2}, R_{v_3} intersect with R_{u_1} such that R_{v_2} lies to the right of R_{v_1} and to the left of R_{v_3} as shown in Figure 3(b). It is easy to observe that $b_{v_3} > b_{v_2} > b_{v_1}$, or else it is not possible to define R_{u_2}, R_{u_3}, and R_{u_4}. Since R_{u_3} has to be defined such that $a_{u_3} > a_{v_1}$ and $b_{u_3} < b_{u_1}$, it is not possible to define R_{v_5} such that it intersects with R_{u_3} but not with R_{u_1}, a contradiction. ■

A path P in a tree T is called a *spine* of T if every vertex of T is within distance two from at least one vertex of P.

Theorem 1: A tree T has a spine if and only if T does not contain 3-claw as a subtree.

Proof: The necessity is obvious. To prove the sufficiency, assume T does not contain a 3-claw. Let P be a longest path in T. We claim that P is a spine. Assume it is not. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p-1$. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in F containing v_i, $1 \leq i \leq p$. Since P is a longest path in T, T_1 consists of only one vertex, v_1, and T_p consists of only one vertex, v_p. Also all vertices in T_2 and T_{p-1} are within distance one from v_2 and v_{p-1}, respectively; and all vertices in T_3 and T_{p-2} are within distance two from v_3 and v_{p-2}, respectively. Since we assumed that P is not a spine, there exists an integer j $(4 \leq j \leq p-3)$ such that T_j contains a vertex w_j whose distance from v_j is three. Let P'



(a) 3-claw. (b) Rays.

Fig. 3.

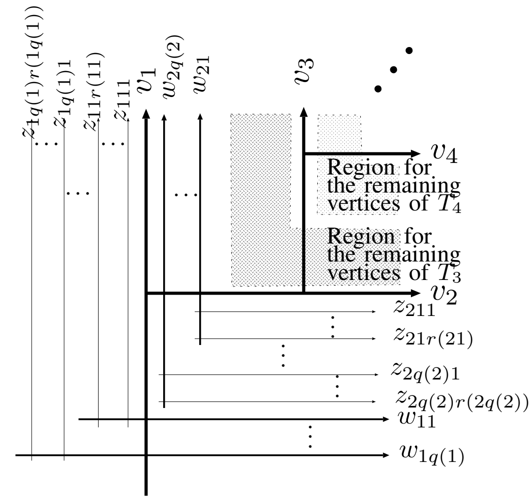


Fig. 4. Rays corresponding to the vertices of orthogonal ray tree T. A ray is labelled with the vertex it corresponds to.

be the path from v_j to w_j. Then the subgraph of T induced by the vertices in $\{v_i \mid j-3 \leq i \leq j+3\} \cup V(P')$ is a 3-claw. This contradicts the assumption that T does not contain 3-claw as a subtree, and therefore P is a spine. ■

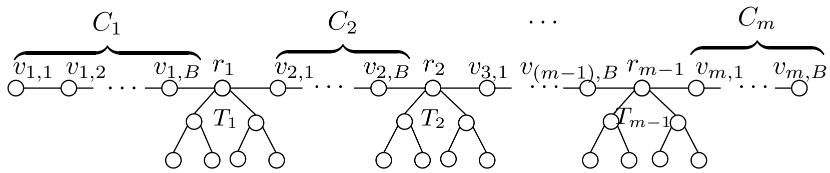
Theorem 2: A tree T is a 2-directional orthogonal ray tree if and only if T does not contain 3-claw as a subtree.

Proof: The necessity follows from Lemma 1. We will show the sufficiency. Assume T does not contain 3-claw as a subtree. Then from Theorem 1, T contains a spine P. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \leq i \leq p-1$. Corresponding to each vertex v_i in P, define ray $R_{v_i} = \{(i, y) \mid y \geq i-1\}$ if i is odd, and define ray $R_{v_i} = \{(x, i) \mid x \geq i-1\}$ if i is even. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in T containing v_i, $1 \leq i \leq p$. Consider T_i to be rooted at v_i. Let $w_{i1}, w_{i2}, \ldots, w_{iq(i)}$ be the children of v_i in T_i, where $q(i)$ is the number of children of v_i in T_i. Let $z_{ij1}, z_{ij2}, \ldots, z_{ijr(ij)}$ be the children of w_{ij} in T_i, where $r(ij)$ is the number of children of w_{ij} in T_i. The rays corresponding to w_{ij} and z_{ijk}, $(1 \leq i \leq p$, $1 \leq j \leq q(i)$, $1 \leq k \leq r(ij))$ can be added as shown in Figure 4. Thus T is a 2-directional orthogonal ray graph. ■

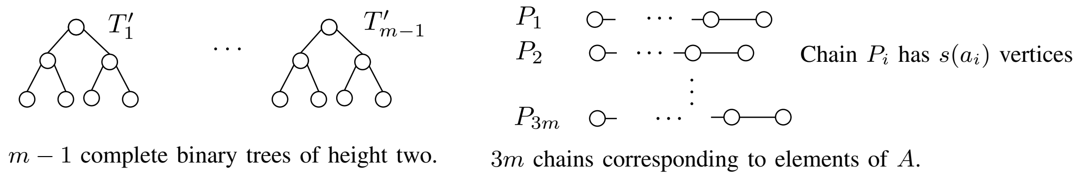
III. Intractability of LOGIC MAPPING

We show in this section the following.

Theorem 3: LOGIC MAPPING is NP-hard.



(a) G.



(b) H.

Fig. 5. Two-directional orthogonal ray tree G and forest H corresponding to the instance of 3-PARTITION.

Theorem 3 follows from Theorem 4 below. A decision problem associated with the subgraph isomorphism problem is stated as follows.

SUBGRAPH ISOMORPHISM

INSTANCE: Graphs H and G.

QUESTION: Does G contain a subgraph isomorphic to H, that is, does there exist a one-to-one mapping $\phi : V(H) \to V(G)$ such that if $(u, v) \in E(H)$ then $(\phi(u), \phi(v)) \in E(G)$?

Theorem 4: SUBGRAPH ISOMORPHISM is NP-complete even if G is a 2-directional orthogonal ray tree and H is a forest.

Proof: It is easy to see that the problem is in NP. We show a polynomial time reduction from 3-PARTITION, which has been shown to be strongly NP-complete in [2]. 3-PARTITION is defined as follows.

3-PARTITION

INSTANCE: A finite set A of $3m$ elements, a bound $B \in \mathbb{Z}^+$, and a size $s(a) \in \mathbb{Z}^+$ for each $a \in A$, such that each $s(a)$ satisfies $B/4 < s(a) < B/2$ and such that $\sum_{a \in A} s(a) = mB$.

QUESTION: Does A have a 3-partition, that is, can A be partitioned into m disjoint sets $S_1, S_2, \ldots, S_m$ such that, for $1 \le i \le m$, $\sum_{a \in S_i} s(a) = B$?

Let $A = \{a_1, a_2, \ldots, a_{3m}\}$, $B \in \mathbb{Z}^+$, and $s(a_1), s(a_2), \ldots, s(a_{3m}) \in \mathbb{Z}^+$ be an instance of 3-PARTITION in which $\max_{a \in A}\{s(a)\}$ is bounded by a polynomial of the size of the instance . We shall construct a 2-directional orthogonal ray tree G and a forest H as follows.

Let $C_1, C_2, \ldots, C_m$ be B-vertex chains such that for each i $(1 \le i \le m)$, $V(C_i) = \{v_{i,j} \mid 1 \le j \le B\}$ and $E(C_i) = \{(v_{i,j}, v_{i,(j+1)}) \mid 1 \le j \le B-1\}$. Let $T_1, T_2, \ldots, T_{m-1}$ be complete binary trees of height two rooted at vertices $r_1, r_2, \ldots, r_{m-1}$, respectively. Let G be the graph defined as

$$\begin{aligned} V(G) &= \left(\bigcup_{i=1}^{m} V(C_i)\right) \cup \left(\bigcup_{i=1}^{m-1} V(T_i)\right), \\ E(G) &= \left(\bigcup_{i=1}^{m} E(C_i)\right) \cup \left(\bigcup_{i=1}^{m-1} E(T_i)\right) \cup \\ & \quad \{(r_i, v_{i,B}), (r_i, v_{(i+1),1}) \mid 1 \le i \le m-1\}. \end{aligned}$$

(See Figure 5(a).) Since the path in G from $v_{1,1}$ to $v_{m,B}$ is a spine of G, it follows from Theorems 1 and 2 that G is a two-directional orthogonal ray tree. Let H be a forest consisting of $m-1$ complete binary trees of height two $T'_1, T'_2, \ldots, T'_{m-1}$, and $3m$ chains $P_1, P_2, \ldots P_{3m}$, each P_j corresponding to element a_j of A and having $s(a_j)$ vertices. (See Figure 5(b).) G and H can be constructed in time polynomial in m and B.

We next prove that A has a 3-partition if and only if G contains a subgraph isomorphic to H.

Suppose first that A can be partitioned into m disjoint subsets $S_1, S_2, \ldots, S_m$ such that for each i $(1 \le i \le m)$, $\sum_{a \in S_i} s(a) = B$. An isomorphism from H to a subgraph of G can be obtained as follows. Since each chain C_i contains B vertices, we can map the chains of H corresponding to the elements of S_i to the chain C_i in G. Each T'_i in H can be mapped to T_i in G. It is easy to see that this is indeed an isomorphism from H to a subgraph of G.

Next suppose that H is isomorphic to a subgraph of G. Each $T'_j (1 \le j \le m-1)$ in H contains two vertices which have degree three and are at a distance two from each other. For a pair of vertices in G, the same is true only if the two vertices are the children of vertex r_i in T_i for any i $(1 \le i \le m-1)$. Therefore, each T'_j in H must be mapped to some T_i in G. This means that chains $P_1, P_2, \ldots, P_{3m}$ in H are mapped to chains $C_1, C_2, \ldots, C_m$ in G. For $1 \le i \le m$, let S_i be the set of elements of A corresponding to the paths of H mapped to C_i. Since C_i has B vertices, $\sum_{a \in S_i} s(a) \le B$,

Input:	A set $\mathcal{H}$ of rightward rays, a set $\mathcal{V}$ of upward rays, and an integer k.
Output:	YES, if $\mathcal{H} \cup \mathcal{V}$ contains a $k \times k$ sub-crossbar. NO, otherwise.
Step 1:	If $\mathcal{H}$ or $\mathcal{V}$ is empty, output NO and halt. Else, set B to be the bottommost ray in $\mathcal{H}$ and set L to be the leftmost ray in $\mathcal{V}$.
Step 2:	Set n_B to be the number of rays in $\mathcal{V}$ that intersect with B, and set n_L to be the number of rays in $\mathcal{H}$ that intersect with L.
Step 3:	If $n_B \geq k$ and $n_L \geq k$, output YES.
Step 4:	If $n_B < k$, set $\mathcal{H} = \mathcal{H} - \{B\}$.
Step 5:	If $n_L < k$, set $\mathcal{V} = \mathcal{V} - \{L\}$.
Step 6:	Return to Step 1.

Fig. 6. Algorithm 1.

for all i $(1 \leq i \leq m)$. Moreover, since the instance of 3-PARTITION satisfies $\sum_{a \in A} s(a) = mB$, we can conclude that $\sum_{a \in S_i} s(a) = B$ for all i $(1 \leq i \leq m)$. Therefore A has a 3-partition. ■

IV. Tractability of SQUARE SUB-CROSSBAR

IV-A. Two-Directional Orthogonal Rays

If we restrict the instance of SQUARE SUB-CROSSBAR such that all horizontal rays are directed towards the right and all vertical rays are directed upwards, we can solve the problem with a simple algorithm outlined in Figure 6, where we consider a decision problem associated with SQUARE SUB-CROSSBAR for simplicity. It is not difficult to see the following:

Theorem 5: Algorithm 1 solves a decision problem associated with SQUARE SUB-CROSSBAR with the instance restricted to righward or upward rays in $O((|\mathcal{H}|+|\mathcal{V}|)^2)$ time.

IV-B. General Orthogonal Rays

We shall next extend Algorithm 1 to cover the case for general orthogonal rays.

Let $\mathcal{R}_X$ be a set of horizontal rays and $\mathcal{R}_Y$ be a set of vertical rays. Suppose two rays $R_x \in \mathcal{R}_X$ and $R_y \in \mathcal{R}_Y$ intersect at point P. Define $\mathcal{R}_Y^{xy} \subseteq \mathcal{R}_Y$ to be the set of rays that intersect with R_x and are to the left of P. Similarly define $\mathcal{R}_X^{xy} \subseteq \mathcal{R}_X$ to be the set of rays that intersect with R_y and are below P. Let (x_L, y_L) be the point where the leftmost ray in $\mathcal{R}_Y^{xy}$ intersects R_x, and let (x_B, y_B) be the point where the bottommost ray in $\mathcal{R}_X^{xy}$ intersects R_y. For each ray $R \in \mathcal{R}_Y^{xy}$ with endpoint (x_R, y_R), define ray V_R as follows: $V_R = R$ if R is an upward ray, and V_R is an upward ray with endpoint (x_R, y_B) if R is a downward ray. And for each ray $R \in \mathcal{R}_X^{xy}$ with endpoint (x_R, y_R), define ray H_R as follows: $H_R = R$ if R is a rightward ray, and H_R is a rightward ray with endpoint (x_L, y_R) if R is a leftward ray. Finally, define $\mathcal{V}^{xy} = \{V_R \mid R \in \mathcal{R}_Y^{xy}\}$, and define $\mathcal{H}^{xy} = \{H_R \mid R \in \mathcal{R}_X^{xy}\}$.

The following observation is obvious from the definitions above.

Observation 1: Two rays in $\mathcal{V}^{xy} \cup \mathcal{H}^{xy}$ intersect if and only if their corresponding rays in $\mathcal{R}_Y^{xy} \cup \mathcal{R}_X^{xy}$ intersect.

Input:	A set $\mathcal{R}_X$ of horizontal rays and a set $\mathcal{R}_Y$ of vertical rays, and a positive integer k.
Output:	YES if $\mathcal{R}_X \cup \mathcal{R}_Y$ contains a $k \times k$ sub-crossbar. NO, otherwise.
Step 1:	Set $S = \{(R_x, R_y) \mid R_x \in \mathcal{R}_X, R_y \in \mathcal{R}_Y$, and R_x and R_y intersect$\}$.
Step 2:	If S is empty, ouput NO and halt. Else arbitrarily choose a pair (R_x, R_y) from S and apply Algorithm 1 with $\mathcal{H}^{xy}$, $\mathcal{V}^{xy}$, and $k-1$ as inputs.
Step 3:	If Algorithm 1 returns YES, output YES.
Step 4:	If Algorithm 1 returns a NO, set $S = S - \{(R_x, R_y)\}$ and return to Step 2.

Fig. 7. Algorithm 2.

Observation 2: $\mathcal{R}_X \cup \mathcal{R}_Y$ contains a $k \times k$ survivig sub-crossbar if and only if there exists a pair of intersecting rays $R_x \in \mathcal{R}_X$ and $R_y \in \mathcal{R}_Y$ such that $\mathcal{H}^{xy} \cup \mathcal{V}^{xy}$ contains a $(k-1) \times (k-1)$ surviving sub-crossbar.

Proof: The sufficiency is immediate from Observation 1. To see the necessity, set R_x and R_y to be the topmost and rightmost rays, respectively of a $k \times k$ sub-crossbar. ■

Figure 7 shows Algorithm 2 which uses Algorithm 1 as a sub-routine.

Algorithm 2 exhaustively checks all pairs of intersecting rays to determine if there exists a pair $R_x \in \mathcal{R}_X$ and $R_y \in \mathcal{R}_Y$ such that $\mathcal{H}^{xy} \cup \mathcal{V}^{xy}$ contains a $(k-1) \times (k-1)$ surviving sub-crossbar. Therefore, from Observation 2 and Theorem 5, we obtain the following.

Theorem 6: Algorithm 2 solves a decision problem associated with SQUARE SUB-CROSSBAR in $O((|\mathcal{R}_X|+|\mathcal{R}_Y|)^4)$ time.

V. Concluding Remarks

It should be noted that Algorithm 2 can be easily modified for the search version and the original optimization version of SQUARE-CROSSBAR. It should also be noted that Algorithm 2 can be used to decide the presence of a $k \times k$ sub-crossbar even if the input sets $\mathcal{R}_X$ and $\mathcal{R}_Y$ contain line segments instead of rays. Moreover, Algorithm 2 can be easily modified to decide the presence of an $m \times n$ sub-crossbar for any positive integers m and n. It is an interesting open question to reduce the complexity of Algorithms 1 and 2.

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