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On orthogonal ray graphs

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ABSTRACT

An orthogonal ray graph is an intersection graph of horizontal and vertical rays (half-lines) in the *xy*-plane. An orthogonal ray graph is a 2-directional orthogonal ray graph if all the horizontal rays extend in the positive *x*-direction and all the vertical rays extend in the positive *y*-direction. We first show that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs. We next provide several characterizations of 2-directional orthogonal ray graphs. Our first characterization is based on forbidden submatrices. A characterization in terms of a vertex ordering follows immediately. Next, we show that 2-directional orthogonal ray graphs are exactly those bipartite graphs whose complements are circular arc graphs. This characterization implies polynomial-time recognition and isomorphism algorithms for 2-directional orthogonal ray graphs. It also leads to a characterization of 2-directional orthogonal ray graphs by a list of forbidden induced subgraphs. We also show a characterization of 2-directional orthogonal ray trees, which implies a linear-time algorithm to recognize such trees. Our results settle an open question of deciding whether a (0, 1)-matrix can be permuted to avoid the submatrices $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$ and $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$.

1. Introduction

A bipartite graph G with a bipartition (U, V) is called an *orthogonal ray graph* if there exist a family of non-intersecting rays (half-lines) R_u, $u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting rays R_v, $v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. An orthogonal ray graph G is called a 2-*directional orthogonal ray graph* if $R_u = \{(x, b_u) \mid x \geq a_u\}$ for each $u \in U$, and $R_v = \{(a_v, y) \mid y \geq b_v\}$ for each $v \in V$, where a_w and b_w are real numbers for any $w \in U \cup V$.

We introduced orthogonal ray graphs [18] in connection with defect tolerance schemes for nano-programmable logic arrays [17,21]. We used orthogonal ray graphs to model defective nano-wire crossbars, with each vertex representing a nano-wire and an edge representing a programmable connection point between nano-wires. Interested readers are referred to [18] for more information.

In this paper, we investigate orthogonal ray graphs, focusing mainly on 2-directional orthogonal ray graphs. We provide several characterizations of 2-directional orthogonal ray graphs and their consequences on the recognition and isomorphism problems of such graphs and an open question posed by Klinz et al. [9].

A bipartite graph G with a bipartition (U, V) is called a *grid intersection graph* if there exist a family of non-intersecting line segments L_u, $u \in U$, parallel to the x-axis in the xy-plane, and a family of non-intersecting line segments L_v, $v \in V$, parallel to the y-axis such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if L_u and L_v intersect. A grid intersection graph is said to be *unit* if all the line segments corresponding to the vertices have the same length. We show in Section 3 that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs.

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Let G be a bipartite graph with a bipartition (U, V). A (0, 1)-matrix $M = [m_{ij}]$ is called a *bipartite adjacency matrix* of G if the rows of M correspond to the vertices of U, the columns of M correspond to the vertices of V, and $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$, where $u_i \in U$ is a vertex corresponding to row i and $v_j \in V$ is a vertex corresponding to column j. Let A, B, and C be matrices. B is said to be a submatrix of A if B can be obtained by deleting some rows and columns of A. A is said to be *C-free* if A does not contain C as a submatrix. For a set S of matrices, A is said to be *S-free* if A is M-free for every $M \in S$. A is said to be *S-freeable* if there exist a permutation of the rows of A and a permutation of the columns of A such that the permuted matrix is S-free. Let

$$\gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

We show in Section 4.1.1 that a bipartite graph G is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of G is γ-freeable.

A bipartite graph G with bipartition (U, V) is said to be *weakly orderable* if there exist an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U and an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that for every i, i', j, j' $(1 \le i < i' \le |U|, 1 \le j < j' \le |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$. We show in Section 4.1.2 that a graph G is a 2-directional orthogonal ray graph if and only if G is weakly orderable.

A graph G is a *circular arc graph* if there exists a collection of circular arcs A_u, $u \in V(G)$ on a fixed circle, such that two arcs A_v and A_w intersect if and only if $(v, w) \in E(G)$. We show in Section 4.1.3 that a bipartite graph G is a 2-directional orthogonal ray graph if and only if the complement of G is a circular arc graph. This characterization implies polynomial-time recognition and isomorphism algorithms for 2-directional orthogonal ray graphs, thereby settling an open question of deciding whether a matrix is γ-freeable raised by Klinz et al. [9].

An *edge-asteroid* is a set of edges $e_0, e_1, \ldots, e_{2k}$ such that for each $i = 0, 1, \ldots, 2k$, there is a path containing e_i and $e_{i+1 (\mathrm{mod}\ 2k+1)}$ that avoids the neighbors of the end-vertices of $e_{i+k+1 (\mathrm{mod}\ 2k+1)}$. We obtain from a result by Feder et al. [5] that a graph G is a 2-directional orthogonal ray graph if and only if it contains no induced cycles of length at least 6 and no edge-asteroids.

Let $(X, \le)$ be a partially ordered set (poset). For $x, y \in X$, we shall use the notation $x < y$ to mean $x \le y$ and $x \ne y$. The *interval dimension* of a poset $(X, \le)$ is the least positive integer k for which there exists a function F which assigns to each $x \in X$, a sequence $\{F(x)(i) : 1 \le i \le k\}$ of k closed intervals on the real line so that $x < y$ if and only if $F(x)(i)$ lies completely to the left of $F(y)(i)$ for all $1 \le i \le k$. A *bipartite poset* is a triple $(X, Y, \le)$ where X and Y are disjoint sets and $\le$ is a partial order on $X \cup Y$ with $\{(x, y) | x < y\} \subseteq X \times Y$. With a bipartite graph G having bipartition (X, Y), we associate a bipartite poset $P_G = (X, Y, \le)$, such that for every pair $(x, y) \in X \times Y$, $x < y$ if and only if $(x, y) \in E(G)$. We obtain from a result by Trotter and Moore [23], that G is a 2-directional orthogonal ray graph if and only if P_G is a bipartite poset of interval dimension at most 2. This connection allows us to characterize 2-directional orthogonal ray graphs by a list of forbidden induced subgraphs, which we show in Section 4.1.4.

The 3-*claw* is a tree obtained from a complete bipartite graph $K_{1,3}$ by replacing each edge with a path of length 3. We show in Section 4.2 that a tree T is a 2-directional orthogonal ray graph if and only if T does not contain 3-claw as a subtree. It follows that we can decide in linear time whether a given tree is a 2-directional orthogonal ray graph.

2. Survey of related graph classes

Grid intersection graphs are well studied. Hartman et al. [6] and Pach et al. [16] independently showed that every planar bipartite graph is a grid intersection graph. Let

$$X = \left\{ \begin{bmatrix} w & 1 & x \\ 1 & 0 & 1 \\ y & 1 & z \end{bmatrix} \middle| \, w, x, y, z \in \{0, 1\} \right\}.$$

Hartman et al. [6] showed that a bipartite graph G is a grid intersection graph if and only if a bipartite adjacency matrix of G is X-freeable. Kratochvíl [11] showed that the recognition problem for grid intersection graphs is NP-complete.

Unit grid intersection graphs were investigated by Otachi et al. [15]. They showed that a bipartite graph G is a unit grid intersection graph if a bipartite adjacency matrix of G is γ-freeable.

A bipartite graph is *chordal bipartite* if it contains no cycle of length at least 6 as an induced subgraph. A graph G is chordal bipartite if and only if a bipartite adjacency matrix of G is Γ-freeable (see for example [9]), where

$$\Gamma = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right\}.$$

Lubiw [12] showed a polynomial-time recognition algorithm for chordal bipartite graphs based on Γ-free matrices.

A graph G with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ is called a *permutation graph* if there exists a pair of permutations π_1 and π_2 on $N = \{1, 2, \ldots, n\}$ such that for all $i, j \in N$, $(v_i, v_j) \in E(G)$ if and only if

$$(\pi_1^{-1}(i) - \pi_1^{-1}(j))(\pi_2^{-1}(i) - \pi_2^{-1}(j)) < 0.$$

A bipartite graph G with bipartition (U, V) is said to be *strongly orderable* if there exist an ordering $(u_1, u_2, \ldots, u_{|U|})$ of U and an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that for any integers i, i', j, j' $(1 \le i < i' \le |U|, 1 \le j < j' \le |V|)$, $(u_i, v_{j'}) \in E(G)$ and $(u_{i'}, v_j) \in E(G)$ imply $(u_i, v_j) \in E(G)$ and $(u_{i'}, v_{j'}) \in E(G)$. Spinrad et al. [20] showed that a bipartite graph G is a permutation graph if and only if G is strongly orderable, and gave a linear-time recognition algorithm for bipartite permutation graphs based on this characterization. Let

$$\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

It also follows from the characterization that a bipartite graph G is a permutation graph if and only if a bipartite adjacency matrix of G is β-freeable as shown by Chen and Yesha [3].

A bipartite graph G with bipartition (U, V) is said to be *convex* if there exists an ordering $(v_1, v_2, \ldots, v_{|V|})$ of V such that, for every $u \in U$ and integers i, j $(1 \le i < j \le |V|)$, $(u, v_i) \in E(G)$ and $(u, v_j) \in E(G)$ imply that $(u, v_k) \in E(G)$ for every integer k $(i \le k \le j)$. It is mentioned in [4] that convex bipartite graphs can be recognized in linear time using PQ-trees.

A bipartite graph G with a bipartition (U, V) is called an *interval bigraph* if every vertex $w \in U \cup V$ can be assigned an interval I_w on the real line so that for all $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if I_u and I_v intersect. The class of interval bigraphs, which properly contains the class of bipartite permutation graphs, has been extensively studied. Müller [14] noted that the class of interval bigraphs is a proper subset of the class of chordal bipartite graphs and provided a polynomial-time recognition algorithm for interval bigraphs. Hell and Huang [7] showed that G is an interval bigraph if and only if the complement of G is a circular arc graph which has a circular arc representation in which no two arcs together cover the whole circle. They showed that an interval bigraph contains no induced cycles of length at least 6 and no edge-asteroids. They also provided a characterization of interval bigraphs in terms of a vertex ordering.

3. Orthogonal ray graphs

The following theorem is implicit in [10], where it is mentioned without a proof. We shall provide a proof sketch.

Theorem 1. *A cycle C_{2n} of length $2n$ is an orthogonal ray graph if and only if $2 \le n \le 6$.*

Proof Sketch. It can be easily verified that C_{2n} is an orthogonal ray graph for $2 \le n \le 6$. We show that C_{2n} is not an orthogonal ray graph for any $n \ge 7$. Suppose to the contrary that C_{2n} is an orthogonal ray graph for some $n \ge 7$. Let $V(C_{2n}) = \{0, 1, \ldots, 2n-1\}$. Let e_i be the edge connecting vertices i and $i + 1 (\mathrm{mod}\ 2n)$, and let $E(C_{2n}) = \{e_i \mid 0 \le i \le 2n\}$. Two edges e_i and e_j of C_{2n} are said to be separable if $|i - j| (\mathrm{mod}\ 2n) \ge 3$. Note that no end-vertex of e_i is adjacent to any end-vertex of e_j if e_i and e_j are separable. Let $E' \subset E(C_{2n})$ be a largest set of mutually separable edges. Note that $|E'| \ge 4$, since $n \ge 7$. For a representation of C_{2n} by orthogonal rays, we can classify the edges of C_{2n} as being one of the four types: UR, DR, UL, or DL, depending on the orientation (upward or downward) of the vertical ray and the orientation (rightward or leftward) of the horizontal ray corresponding to the end-vertices of the edge.

We first show that in any representation of C_{2n} by orthogonal rays, no two edges of E' can be of the same type. Assume the contrary, and suppose that for some $e_i, e_j \in E'$, e_i and e_j are of the same type. Without loss of generality, let e_i and e_j be of the UR type. For a UR edge e, we call the region above the horizontal ray and right of the vertical ray as the "inner" region of e, and the remaining region as the "outer" region. Since e_i and e_j are separable, either the rays corresponding to e_i lie entirely in the inner region of e_j or vice versa. Again without loss of generality, let us assume the former. Consider a third edge $e_k \in E'$. We can identify three regions where the rays corresponding to the end-vertices of e_k can lie: (i) inner region of e_i, (ii) outer region of e_j, or (iii) the intersection of the inner region of e_j and outer region of e_i. We shall reach a contradiction to the assumption that e_i and e_j are of the same type by showing that e_k cannot lie in any of the regions. Consider the first region. Since there is a path in C_{2n} connecting e_i and e_k that avoids the neighbors of the end-vertices of e_j, e_k cannot lie in the first region. The other two cases follow similarly.

For any $n \ge 8$, we have $|E'| \ge 5$, whereas an edge can be one of only 4 types. This leads to a contradiction to the assumption that $C_{2n>16}$ is an orthogonal ray graph. For $n = 7$, we have $|E'| = 4$, and therefore each edge in E' must be of a different type. Even in this case, we can arrive at a contradiction; however, the proof is rather tedious, and we shall omit the details. □

Theorem 2. *The class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs.*

Proof. Let G be an orthogonal ray graph with bipartition (U, V). Let S be a square on the xy-plane such that all the cross points of rays R_w, $w \in U \cup V$, lie in S. Let l be the length of a side of S. Each ray intersects with at least one side of S. If a ray R_w intersects with both the opposite sides of S, let L_w be the line segment such that the endpoints of L_w are the two crossing points where R_w intersects with the sides of S. If a ray R_w intersects with only one side of S, let L_w be the line segment such that the endpoints of L_w are the endpoint of R_w and the point on R_w at a distance l from the endpoint of R_w. Since all the line segments have length l, and L_u and L_v intersect if and only if R_u and R_v intersect, G is a unit grid intersection graph for line segments $\{L_u \mid u \in U\} \cup \{L_v \mid v \in V\}$. Thus the class of orthogonal ray graphs is a subset of the class of unit grid intersection graphs.

It is easy to see that C_{2n} is a unit grid intersection graph for any $n \ge 2$. Thus we conclude by Theorem 1 that the class of orthogonal ray graphs is a proper subset of the class of unit grid intersection graphs. □

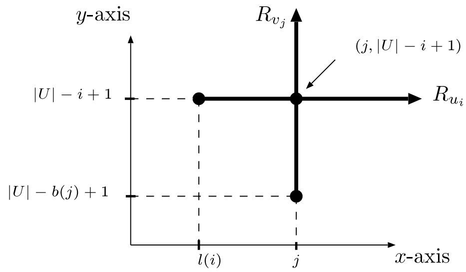

Fig. 1. Rays R_{u_i} and R_{v_j} intersect if and only if $l(i) \leq j$ and $b(j) \geq i$.

4. Two-directional orthogonal ray graphs

4.1. Characterizations of two-directional orthogonal ray graphs

In this section, we give several characterizations of 2-directional orthogonal ray graphs.

4.1.1. Bipartite adjacency matrix characterization

The following is obvious from the definition of γ.

Lemma 3. *An $m \times n$ matrix $M = [m_{ij}]$ is γ-free if and only if for any integers i, i', j, j' $(1 \leq i < i' \leq m, 1 \leq j' < j \leq n)$, $m_{ij'} = 1$ and $m_{i'j} = 1$ imply $m_{ij} = 1$.* □

We can characterize the 2-directional orthogonal ray graphs as follows.

Theorem 4. *A bipartite graph G is a 2-directional orthogonal ray graph if and only if a bipartite adjacency matrix of G is γ-freeable.*

Proof. Let G be a bipartite graph with a bipartition (U, V). Suppose that a bipartite adjacency matrix of G is γ-freeable, and let $M = [m_{ij}]$ be a bipartite adjacency matrix of G which is γ-free. We denote by $u_i \in U$ the vertex corresponding to row i, and by $v_j \in V$ the vertex corresponding to column j. For each row i of M, define $l(i)$ to be the column which contains the leftmost 1 in that row. Then define ray R_{u_i} to be a rightward ray with endpoint at $(l(i), |U| - i + 1)$. Similarly for each column j, define $b(j)$ to be the row which contains the bottommost 1 in that column. Define ray R_{v_j} to be an upward ray with endpoint at $(j, |U| - b(j) + 1)$. Note that from these definitions, two rays R_{u_i} and R_{v_j} intersect if and only if $l(i) \leq j$ and $b(j) \geq i$. (See Fig. 1.) We are now ready to show that R_{u_i} and R_{v_j} intersect if and only if $(u_i, v_j) \in E(G)$. Suppose first that $(u_i, v_j) \in E(G)$. Then $m_{ij} = 1$, which means that $l(i) \leq j$ and $b(j) \geq i$. Therefore, rays R_{u_i} and R_{v_j} intersect. Suppose next that $(u_i, v_j) \notin E(G)$. Then $m_{ij} = 0$. Since M is γ-free, we have $m_{i'j} = 0$ for every $i' > i$ or $m_{ij'} = 0$ for every $j' < j$, by Lemma 3. This means that $l(i) > j$ or $b(j) < i$, which implies that R_{u_i} and R_{v_j} do not intersect. Thus we conclude that G is a 2-directional orthogonal ray graph for rays $\{R_{u_i} | u_i \in U\} \cup \{R_{v_j} | v_j \in V\}$.

Conversely, suppose that G is a 2-directional orthogonal ray graph, and $\{R_u \mid u \in U\} \cup \{R_v \mid v \in V\}$ is the set of rays corresponding to the vertices. Let $(u_1, u_2, \ldots, u_{|U|})$ be the ordering of U such that for any integers i, i' $(1 \leq i < i' \leq |U|)$, R_{u_i} is above $R_{u_{i'}}$ in the xy-plane. Similarly, let $(v_1, v_2, \ldots, v_{|V|})$ be the ordering of V such that for any integers j, j' $(1 \leq j < j' \leq |V|)$, R_{v_j} is to the left of $R_{v_{j'}}$. Construct a bipartite adjacency matrix $M = [m_{ij}]$ of G such that $m_{ij} = 1$ if and only if $(u_i, v_j) \in E(G)$. We shall show that M is γ-free. For some integers i, i', j, j', $(1 \leq i < i' \leq |U|, 1 \leq j' < j \leq |V|)$, suppose $m_{i'j} = 1$ and $m_{ij'} = 1$. Since ray R_{u_i} is above ray $R_{u_{i'}}$ and $R_{v_{j'}}$ is to the left of R_{v_j}, R_{u_i} must intersect with R_{v_j} implying that $m_{ij} = 1$. Thus from Lemma 3, M is γ-free. □

4.1.2. Vertex order characterization

The following corollary is immediate from Theorem 4.

Corollary 5. *A bipartite graph G is a 2-directional orthogonal ray graph if and only if G is weakly orderable.*

Proof. Let M be a bipartite adjacency matrix of a bipartite graph G with bipartition (U, V). Consider the ordering $(u_1, u_2, \ldots, u_{|U|})$ of U such that u_i is the vertex corresponding to row i of M, and the ordering $(v_{|V|}, v_{|V|-1}, \ldots, v_1)$ of V, such that v_j is the vertex corresponding to column j. It is easy to see that these orderings of U and V form a weak ordering of G if and only if M is a γ-free matrix, and thus we have the corollary. □

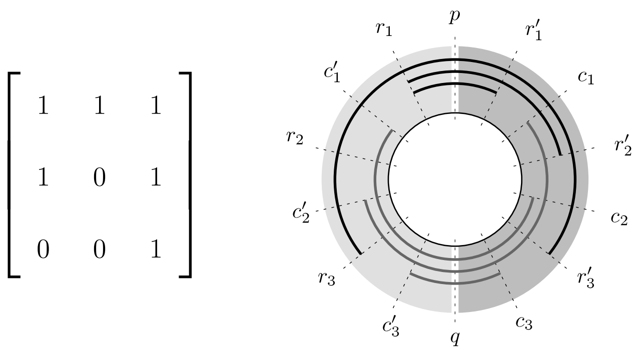

Fig. 2. An example of a family of circular arcs corresponding to a γ-free bipartite adjacency matrix.

4.1.3. Characterization in terms of circular arc graphs

An arc A on a circle O can be denoted by a pair of its endpoints $(s(A), t(A))$, where A is obtained by traversing O clockwise from its counterclockwise endpoint $s(A)$ to its clockwise endpoint $t(A)$.

Lemma 6. *The complement of a* 2*-directional orthogonal ray graph is a circular arc graph.*

Proof. Suppose a bipartite graph G with bipartition (U, V) is a 2-directional orthogonal ray graph. G has a γ-free bipartite adjacency matrix $M = [m_{ij}]$, by Theorem 4. For each row $i(1 \le i \le |U|)$ of M, define $l(i)$ to be the column which contains the leftmost 1 in that row, and for each column $j(1 \le j \le |V|)$, define $b(j)$ to be the row which contains the bottommost 1 in that column. Let O be a circle and let

$$p, r_1', c_1, r_2', c_2, \ldots, r_{|U|}', c_{|U|}, q, c_{|V|}', r_{|V|}, c_{|V-1|}', r_{|V-1|}, \ldots, c_1', r_1 \tag{1}$$

be $2|U| + 2|V| + 2$ distinct points on O in the order of their occurrence in a clockwise traversal of O starting from p. Corresponding to each row i, define arc R_i to be $(r_{l(i)}, r_i')$ and corresponding to each column j, define arc C_j to be $(c_{b(j)}, c_j')$. (An example is shown in Fig. 2.) We shall now show that two arcs R_i and C_j intersect if and only if $m_{ij} = 0$. Suppose first that $m_{ij} = 1$, which implies $i \le b(j)$ and $l(i) \le j$. Since $i \le b(j)$, we can see that r_i' precedes $c_{b(j)}$ in Sequence (1). Since we have defined the clockwise endpoint of R_i to be r_i' and the counterclockwise endpoint of C_j to be $c_{b(j)}$, we can deduce that they do not intersect on arc (p, q). Similarly, we can show that $l(i) \le j$ implies R_i and C_j do not intersect on arc (q, p) either. Next suppose $m_{ij} = 0$. Since M is γ-free, we have $m_{i'j} = 0$ for every $i' > i$ or $m_{ij'} = 0$ for every $j' < j$, by Lemma 3. This means that $l(i) > j$ or $b(j) < i$. Then from Sequence (1), we can see that both R_i and C_j contain the arc $(r_{l(i)}, c'(j))$ or the arc $(c_{b(j)}, r'(i))$. Finally, all R_i intersect at p, and all C_j intersect at q, and therefore we can conclude that the complement of G is a circular arc graph for the family of arcs $\{R_i | 1 \le i \le |U|\} \cup \{C_j | 1 \le j \le |V|\}$. □

Spinrad [19] showed the following.

Lemma 7. *For a circular arc graph G that can be partitioned into cliques U and V, there exist two points p, q on a circle and a representation by arcs A_w, $w \in V(G)$ on the same circle such that for every $u \in U$, A_u contains p but not q and A_v contains q but not p.* □

Lemma 8. *A bipartite graph is a* 2*-directional orthogonal ray graph if its complement is a circular arc graph.*

Proof. Let G be a bipartite graph with bipartition (U, V). Suppose $\overline{G}$, the complement of G, is a circular arc graph. Let p and q be two points on a circle O, and let $\mathcal{R}_U$ and $\mathcal{C}_V$ be the set of arcs on O corresponding to the vertices in U and V, respectively, such that all arcs in $\mathcal{R}_U$ contain p but not q, and all arcs in $\mathcal{C}_V$ contain q but not p, by Lemma 7. Let $R_1, R_2, \ldots, R_{|U|}$ be the arcs in $\mathcal{R}_U$ in the order of the occurrence of their clockwise endpoints when moving around O in the clockwise direction starting from p, and let $C_1, C_2, \ldots, C_{|V|}$ be the arcs in $\mathcal{C}_V$ in the order of the occurrence of their clockwise endpoints when moving around C in the counterclockwise direction starting from p. Let $M = [m_{ij}]$ be a $|U| \times |V|$ (0, 1)-matrix defined as $m_{ij} = 1$ if and only if R_i and C_j do not intersect. Obviously, M is a bipartite adjacency matrix of G. We shall show that M is γ-free. For some integers i, i', j, j', $(1 \le i < i' \le |U|, 1 \le j' < j \le |V|)$, suppose $m_{i'j} = 1$ and $m_{ij'} = 1$. From the definition of M, $m_{i'j} = 1$ means that $R_{i'}$ and C_j do not intersect. Since they do not intersect, $t(R_{i'})$,the clockwise endpoint of $R_{i'}$, must be counterclockwise from $s(C_j)$, the counterclockwise endpoint of C_j. (See Fig. 3). Also, since $i < i'$, $t(R_i)$ must be counterclockwise from $t(R_{i'})$, and therefore R_i and C_j do not intersect on arc (p, q). Similarly we can show that $m_{ij'} = 1$ implies that R_i and C_j do not intersect on arc (q, p) either. Since R_i and C_j do not intersect anywhere on O, the corresponding matrix entry m_{ij} is 1. Therefore, M is γ-free by Lemma 3, and thus G is a 2-directional orthogonal ray graph, by Theorem 4. □

From Lemmas 6 and 8, we have the following.

Theorem 9. *A bipartite graph G is a* 2*-directional orthogonal ray graph if and only if its complement is a circular arc graph.* □

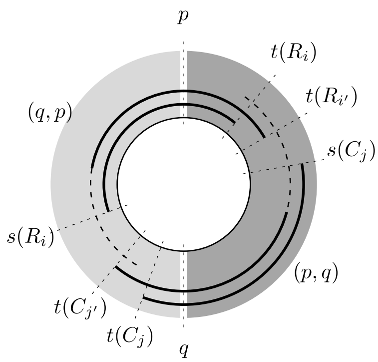

Fig. 3. Arcs R_i, $R_{i'}$, C_j, and $C_{j'}$.

Theorem 9 leads to some interesting consequences as follows. Since McConnell [13] showed a linear-time recognition algorithm for circular arc graphs, we have the following.

Theorem 10. *It can be decided in $O(n^2)$ time whether an n-vertex graph is a 2-directional orthogonal ray graph.* □

From Theorems 4 and 10, we have the following theorem which settles the open problem of recognizing γ-freeable matrices [9].

Theorem 11. *It can be decided in $O((m+n)^2)$ time whether an $m \times n$ matrix is γ-freeable.* □

Feder et al. [5] showed the following:

Theorem 12. *A graph G which can be partitioned into two cliques is a circular arc graph if and only if the complement of G contains no induced cycles of length at least 6 and no edge-asteroids.* □

We recall that a bipartite graph is chordal bipartite if it contains no induced cycles of length at least 6. Then from Theorems 9 and 12, we have

Corollary 13. *A bipartite graph G is a 2-directional orthogonal ray graph if and only if G is chordal bipartite and contains no edge-asteroids.* □

Since Hsu [8] showed that graph isomorphism can be solved in $O(mn)$ time for n-vertex m-edge circular arc graphs, we have the following.

Corollary 14. *The graph isomorphism problem can be solved in $O(n^3)$ time for n-vertex 2-directional orthogonal ray graphs.* □

On the other hand, Uehara et al. [24] showed that the isomorphism problem is GI-complete for chordal bipartite graphs. Thus the class of 2-directional orthogonal ray graphs is a boundary case for the complexity of graph isomorphism. This is an improvement from the earlier boundary class, the interval bigraphs, which is a proper subset of the class of 2-directional orthogonal ray graphs, as we shall show in Section 5.

4.1.4. Forbidden subgraph characterization

Trotter and Moore [23] showed the following.

Theorem 15. *Let G be a graph which can be partitioned into two cliques and let G^c be its complement. Then G is a circular arc graph if and only if the interval dimension of the associated bipartite poset P_{G^c} is at most 2.* □

From Theorems 9 and 15, we obtain the following.

Theorem 16. *A graph G is a 2-directional orthogonal ray graph if and only if the interval dimension of the associated bipartite poset P_G is at most 2.* □

Trotter and Moore [22,23] provided the minimum list $\mathcal{P}$ of posets so that a bipartite poset has interval dimension at most two if and only if it does not contain a poset from $\mathcal{P}$ as a subposet. It is straightforward to derive from $\mathcal{P}$ the minimal list of forbidden induced subgraphs for 2-directional orthogonal ray graphs. This list shown in Fig. 6 contains 6 infinite families of graphs and 3 odd examples.

Theorem 17. *A graph G is a 2-directional orthogonal ray graph if and only if G does not contain any graph in Fig. 6 as an induced subgraph.* □

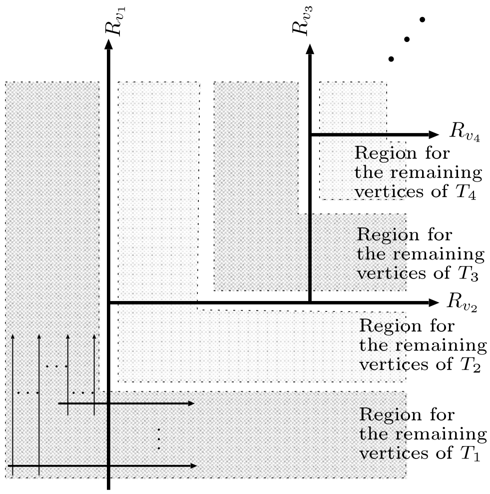

Fig. 4. Rays corresponding to the vertices of T. Rays $R_{v_1}, R_{v_2}, \ldots$ correspond to vertices $v_1, v_2, \ldots$ of the spine, respectively. The remaining vertices of T_i $(1 \le i \le p)$ can be placed in the specified region. To avoid cluttering, only the rays corresponding to the remaining vertices of T_1 are shown.

4.2. Two-directional orthogonal ray trees

Although the characterization of 2-directional orthogonal ray trees we present in this section can be observed from the list of forbidden induced subgraphs in Fig. 6, we provide a direct proof as it suggests a linear-time recognition algorithm for such trees.

A path P in a tree T is called a *spine* of T if every vertex of T is within distance two from at least one vertex of P.

Lemma 18. *A tree T has a spine if and only if T does not contain* 3*-claw as a subtree.*

Proof. The necessity is obvious. To prove the sufficiency, assume T does not contain a 3-claw. Let P be a longest path in T. We claim that P is a spine. Assume it is not. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \le i \le p-1$. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be the tree in F containing v_i, $1 \le i \le p$. Since P is a longest path in T, T_1 consists of only one vertex, v_1, and T_p consists of only one vertex, v_p. Also all vertices in T_2 and T_{p-1} are within distance one from v_2 and v_{p-1}, respectively; and all vertices in T_3 and T_{p-2} are within distance two from v_3 and v_{p-2}, respectively. Since we assumed that P is not a spine, there exists an integer j $(4 \le j \le p-3)$ such that T_j contains a vertex w_j whose distance from v_j is three. Let P' be the path from v_j to w_j. Then the subgraph of T induced by the vertices in $\{v_i \mid j-3 \le i \le j+3\} \cup V(P')$ is a 3-claw. This contradicts the assumption that T does not contain 3-claw as a subtree, and therefore P is a spine. $\square$

Theorem 19. *A tree T is a* 2*-directional orthogonal ray tree if and only if T does not contain* 3*-claw as a subtree.*

Proof. It is easy to see that the 3-claw contains an edge-asteroid, and therefore the necessity follows from Corollary 13. We will show the sufficiency. Assume T does not contain 3-claw as a subtree. Then from Lemma 18, T contains a spine P. Let $V(P) = \{v_1, v_2, \ldots, v_p\}$, and $(v_i, v_{i+1}) \in E(P)$, $1 \le i \le p-1$. Corresponding to each vertex v_i in P, define ray $R_{v_i} = \{(i, y) \mid y \ge i-1\}$ if i is odd, and define ray $R_{v_i} = \{(x, i) \mid x \ge i-1\}$ if i is even. Let F be a forest obtained from T by deleting the edges in $E(P)$. Let T_i be a tree in T containing v_i, $1 \le i \le p$. The rays corresponding to the remaining vertices of T_i can be defined as shown in Fig. 4. Thus T is a 2-directional orthogonal ray graph. $\square$

From Theorem 19 and the proof of Lemma 18, we can see that in order to decide if a given tree T is a 2-directional orthogonal ray graph, we need only to verify whether or not a longest path in T is a spine of T. Since a longest path in a tree can be obtained in linear time (see [1], for example), we can recognize 2-directional orthogonal ray trees in linear time.

5. Class hierarchy

In this section, we explore the relation among the classes of orthogonal ray graphs, 2-directional orthogonal ray graphs, and the graph classes mentioned in Section 2.

From Theorem 1 and Corollary 13, we have the following.

Observation 20. *The class of* 2*-directional orthogonal ray graphs is a proper subset of the class of orthogonal ray graphs.* $\square$

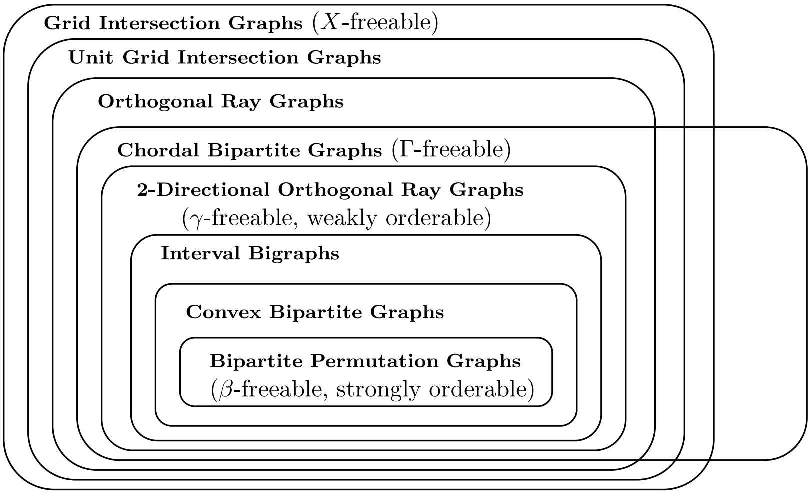

Fig. 5. Relationship between various graph classes.

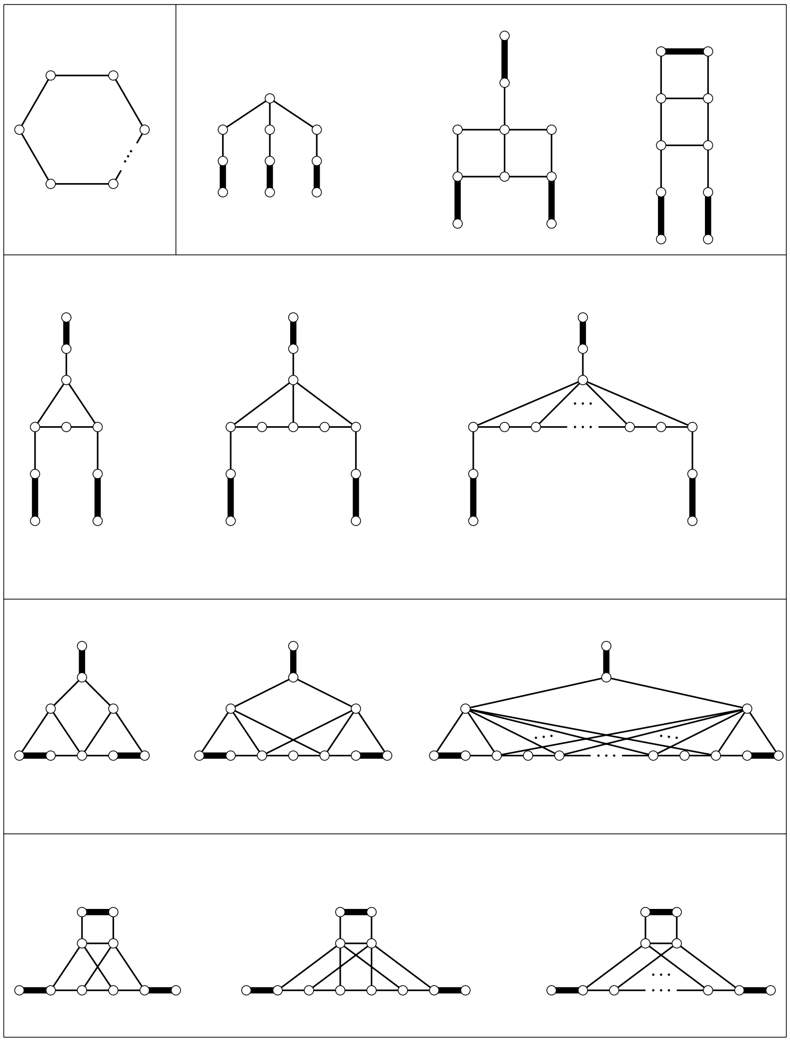

Fig. 6. Forbidden subgraphs for 2-directional orthogonal ray graphs. Bold edges constitute an edge-asteroid.

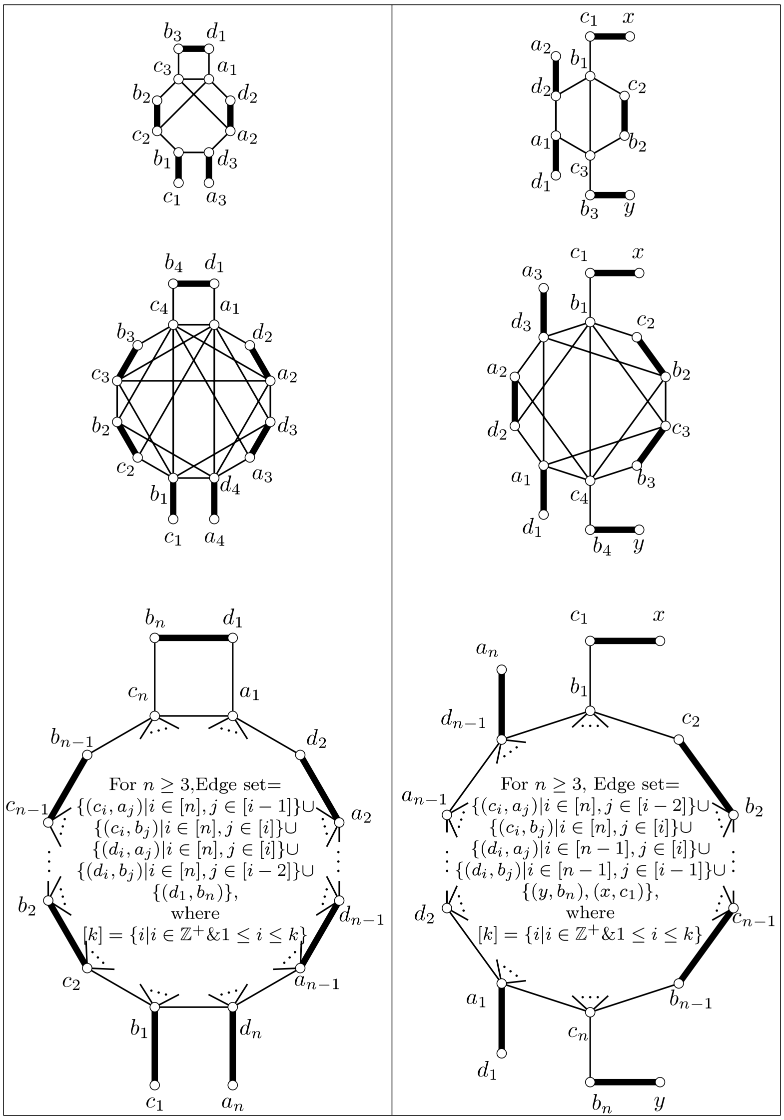

Fig. 6. (*continued*)

Otachi et al. [15] showed that the class of graphs which have a γ-freeable bipartite adjacency matrix properly contains the class of interval bigraphs, and therefore we have the following.

Observation 21. *The class of interval bigraphs is a proper subset of the class of* 2*-directional orthogonal ray graphs.* □

The relationship between the various graph classes mentioned in this paper can be summarized as shown in Fig. 5. It was recently shown by Chandran et al. [2] that the class of chordal bipartite graphs and the class of grid intersection graphs are not comparable.

We conclude by noting that characterization and recognition of orthogonal ray graphs remain open.

References

[1] R.W. Bulterman, F.W. van der Sommen, G. Zwaan, T. Verhoeff, A.J.M. van Gasteren, W.H.J. Feijen, On computing a longest path in a tree, Information Processing Letters 81 (2) (2002) 93–96.
[2] L.S. Chandran, M. Francis, R. Mathew, Chordal bipartite graphs with high boxicity, in: Japan Conference on Computational Geometry and Graphs, 2009, pp. 95–96.
[3] L. Chen, Y. Yesha, Efficient parallel algorithms for bipartite permutation graphs, Networks 23 (1993) 29–39.
[4] P. Damaschke, H. Müller, D. Kratsch, Domination in convex and chordal bipartite graphs, Information Processing Letters 36 (5) (1990) 231–236.
[5] T. Feder, P. Hell, J. Huang, List homomorphisms and circular arc graphs, Combinatorica 19 (1999) 487–505.

[6] I.B.-A. Hartman, I. Newman, R. Ziv, On grid intersection graphs, Discrete Mathematics 87 (1991) 41–52.
[7] P. Hell, J. Huang, Interval bigraphs and circular arc graphs, Journal of Graph Theory 46 (4) (2004) 313–327.
[8] W.-L. Hsu, $O(m.n)$ algorithms for the recognition and isomorphism problems on circular-arc graphs, SIAM Journal on Computing 24 (3) (1995) 411–439.
[9] B. Klinz, R. Rudolf, G.J. Woeginger, Permuting matrices to avoid forbidden submatrices, Discrete Applied Mathematics 60 (1995) 223–248.
[10] A.V. Kostochka, J. Nešetřil, Coloring relatives of intervals on the plane, I: chromatic number versus girth, European Journal of Combinatorics 19 (1998) 103–110.
[11] J. Kratochvíl, A special planar satisfiability problem and a consequence of its NP-completeness, Discrete Applied Mathematics 52 (1994) 233–252.
[12] A. Lubiw, Doubly lexical orderings of matrices, SIAM Journal on Computing 16 (1987) 854–879.
[13] R. McConnell, Linear-time recognition of circular-arc graphs, Algorithmica 37 (2) (2003) 93–147.
[14] H. Müller, Recognizing interval digraphs and interval bigraphs in polynomial time, Discrete Applied Mathematics 78 (1–3) (1997) 189–205.
[15] Y. Otachi, Y. Okamoto, K. Yamazaki, Relationships between the class of unit grid intersection graphs and other classes of bipartite graphs, Discrete Applied Mathematics 155 (2007) 2383–2390.
[16] J. Pach, H. de Fraysseix, P. de Mendez, Representation of planar graphs by segments, Colloquia Mathematica Societatis János Bolyai 63 (1994) 109–117.
[17] W. Rao, A. Orailoglu, A. Karri, Logic mapping in crossbar-based nanoarchitectures, IEEE-Design and Test of Computers 26 (1) (2009) 68–77.
[18] A.M.S. Shrestha, S. Tayu, S. Ueno, Orthogonal ray graphs and nano-pla design, Proceedings of the IEEE International Symposium on Circuits and Systems (2009) 2930–2933.
[19] J. Spinrad, Circular-arc graphs with clique cover number two, Journal of Combinatorial Theory, Series A 44 (3) (1987) 300–306.
[20] J. Spinrad, A. Brandstädt, L. Stewart, Bipartite permutation graphs, Discrete Applied Mathematics 18 (1987) 279–292.
[21] M.B. Tahoori, A mapping algorithm for defect-tolerance of reconfigurable nano-architectures, IEEE/ACM International Conference on Computer-Aided Design (2005) 668–672.
[22] W. Trotter, Combinatorics and Partially Ordered Sets: Dimension Theory, The Johns Hopkins University Press, 1992.
[23] W.T. Trotter, J.I. Moore, Characterization problems for graphs, partially ordered sets, lattices, and families of sets, Discrete Mathematics 16 (1976) 361–381.
[24] R. Uehara, S. Toda, T. Nagoya, Graph isomorphism completeness for chordal bipartite graphs and strongly chordal graphs, Discrete Applied Mathematics 145 (3) (2005) 479–482.