On the Three-Dimensional Orthogonal Drawing of Series-Parallel Graphs (Extended Abstract)

Satoshi Tayu, Kumiko Nomura, and Shuichi Ueno
Department of Communications and Integrated Systems, Tokyo Institute of Technology, Tokyo 152-8550-S3-57, Japan

***Abstract*— It has been known that every 6-graph has a 3-bend 3-D orthogonal drawing, while it has been open whether every 6-graph has a 2-bend 3-D orthogonal drawing. For the interesting open question, it is known that every 5-graph has a 2-bend 3-D orthogonal drawing, and every outerplanar 6-graph without triangles has a 0-bend 3-D orthogonal drawing. We show in this paper that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing.**

I. Introduction

We consider the problem of generating orthogonal drawings of graphs in the space. The problem has obvious applications in the design of 3-D VLSI circuits and optoelectronic integrated systems [7], [11].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of a vertex of G. G is called a k-graph if $\Delta(G) \leq k$. It is well-known that every graph can be drawn in the space so that its edges intersect only at their ends. Such a drawing of a graph G is called a *3-D drawing* of G. A *3-D orthogonal drawing* of G is a 3-D drawing such that each edge is drawn by a sequence of contiguous axis-parallel line segments. Notice that a graph G has a 3-D orthogonal drawing only if $\Delta(G) \leq 6$. A 3-D orthogonal drawing with no more than b bends per edge is called a *b-bend 3-D orthogonal drawing*.

Eades, Symvonis, and Whitesides [4], and Papakostas and Tollis [10] showed that every 6-graph has a 3-bend 3-D orthogonal drawing. Eades, Symvonis, and Whitesides [4] also posed an interesting open question of whether every 6-graph has a 2-bend 3-D orthogonal drawing. Wood [14] showed that every 5-graph has a 2-bend 3-D orthogonal drawing. Nomura, Tayu, and Ueno [9] showed that every outerplanar 6-graph has a 0-bend 3-D orthogonal drawing if and only if it contains no triangle as a subgraph, while Eades, Stirk, and Whitesides [3] proved that it is $\mathcal{NP}$-complete to decide if a given 5-graph has a 0-bend 3-D orthogonal drawing.

We show in this paper the following theorem.

Theorem 1: Every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing. ■

The proof of Theorem 1 is constructive and provides a polynomial time algorithm to generate such a drawing for a series-parallel 6-graph.

It is still open whether every 6-graph has a 2-bend 3-D orthogonal drawing. It is also open whether every series-parallel 6-graph has a 1-bend 3-D orthogonal drawing.

For the two-dimensional case, Biedl and Kant [2], and Liu, Morgana, and Simeone [8] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing with only exception of the octahedron. Moreover, Kant [6] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception of K_4. Tayu, Nomura, and Ueno [12] showed that every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing. Nomura, Tayu, and Ueno [9] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia [5] proved that it is $\mathcal{NP}$-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing. Battista, Liotta, and Vargiu [1] showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs.

II. Series-Parallel Graphs

A *series-parallel graph* is defined recursively as follows.

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the *terminals*.

(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:
   - (i) *Series-composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.
   - (ii) *Parallel-composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into t. Vertices s and t are the terminals of G.

III. 3-D Embeddings and Orthogonal Drawings

The three-dimensional (3-D) grid $\mathcal{G}$ is an (infinite) graph consisting of $\mathbb{Z}^3$, the set of grid-points in 3-D space with integer coordinates, together with the axis-parallel edges connecting neighboring grid-points. The grid-points are also considered as vectors. The *3-D embedding* $\langle \phi, \rho \rangle$ of a graph G is defined by a one-to-one mapping $\phi : V(G) \to V(\mathcal{G}) = \mathbb{Z}^3$, together with a mapping ρ that maps each edge $(u, v) \in E(G)$ onto a path $\rho(u, v)$ in $\mathcal{G}$ that connects $\phi(u)$ and $\phi(v)$. A path P in $\mathcal{G}$ is called a *k-bend path* if P contains k bends.

Let $\mathcal{D}^+ = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$, $\mathcal{D}^- = \{(-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$, and $\mathcal{D} = \mathcal{D}^+ \cup \mathcal{D}^-$. A vector in $\mathcal{D}$ is called a direction.

Let $\langle \phi, \rho \rangle$ be a 3-D embedding of a graph G, and $(u, v) \in E(G)$. If g is a grid-point adjacent with $\phi(u)$ in path $\rho(u, v)$ in $\mathcal{G}$, there exists a direction $\boldsymbol{d} \in \mathcal{D}$ such that $g = \phi(u) + \boldsymbol{d}$.

978-1-4244-1684-4/08/$25.00 ©2008 IEEE

We denote such $\boldsymbol{d}$ by $\boldsymbol{\alpha}_u^\rho(e)$. It is easy to see the following lemma.

Lemma 1: If $\rho(u,v)$ is a 2-bend path, $\rho(u,v)$ is uniquely determined by $\phi(u)$, $\phi(v)$, $\boldsymbol{\alpha}_u^\rho(e)$, and $\boldsymbol{\alpha}_v^\rho(e)$. ■

Figure 1 shows a 2-bend path $\rho(u,v)$ determined by $\phi(u) = (0,0,0)$, $\phi(v) = (3,2,1)$, $\boldsymbol{\alpha}_u^\rho(e) = (0,0,1)$, and $\boldsymbol{\alpha}_v^\rho(e) = (-1,0,0)$.

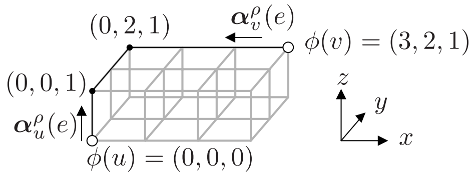


Fig. 1. Example of a 2-bend path $\rho(u,v)$.

Two grid-points g and $g+(a,b,c)$ are said to be in the *general position* if $abc \neq 0$. Let g_0, $g_1 = g_0 + (x_1,y_1,z_1)$, and $g_2 = g_0 + (x_2,y_2,z_2)$ be grid-points in the general position. Then, we define $g_1 \preceq_{g_0} g_2$ if $|w_1| < |w_2|$ or $w_1 w_2 < 0$ for any $w \in \{x,y,z\}$. A 3-D embedding $\langle \phi, \rho \rangle$ of G is called a τ-embedding if all of the following conditions are satisfied:

Condition A: If $u \neq v$ then $\phi(u)$ and $\phi(v)$ are in the general position,

Condition B: For any distinct edges e, $e' \in E(G)$ incident to a vertex u, $\boldsymbol{\alpha}_u^\rho(e) \neq \boldsymbol{\alpha}_u^\rho(e')$,

Condition C: $\rho(e)$ is a 2-bend path for any $e \in E(G)$,

Condition D: For any edges $e_1 = (u,v_1)$ and $e_2 = (u,v_2)$ incident to a vertex u, (D-1) or (D-2) below holds;

(D-1) $\boldsymbol{\alpha}_u^\rho(e_1) = \pm\boldsymbol{\alpha}_{v_2}^\rho(e_2)$;

(D-2) $\phi(v_1) \preceq_{\phi(u)} \phi(v_2)$ or $\phi(v_2) \preceq_{\phi(u)} \phi(v_1)$.

It follows from Condition B that G has a τ-embedding only if G is a 6-graph. The purpose of this section is to show the following theorem.

Theorem 2: A τ-embedding of a 6-graph G induces a 2-bend 3-D orthogonal drawing of G.

Proof (Sketch): The theorem is proved by Lemma 1 together with the following two lemmas.

Lemma 2: Let $\langle \phi, \rho \rangle$ be a 3-D embedding of G satisfying Conditions A through C. If edges e_1 and e_2 have no common ends, paths $\rho(e_1)$ and $\rho(e_2)$ are disjoint.

Proof: Omitted in the extended abstract.

Lemma 3: Let $\langle \phi, \rho \rangle$ be a 3-D embedding of G satisfying Conditions A through C. If any adjacent edges $e_1 = (u,v_1)$ and $e_2 = (u,v_2)$ satisfy Condition D, $\rho(e_1)$ and $\rho(e_2)$ are internally disjoint.

Proof: Omitted in the extended abstract. ■

IV. Proof of Theorem 1 (Sketch)

Let G be a series-parallel 6-graph with terminals s and t. Before proving the theorem, we need some preliminaries.

IV-A. 9-Cubic 3-D Embeddings

Let $p = (p_x,p_y,p_z)$ and $q = (q_x,q_y,q_z)$ be grid-points in the general position, and let $g_{\min}^{(w)}(p,q) = \min\{p_w,q_w\}$ and $g_{\max}^{(w)}(p,q) = \max\{p_w,q_w\}$ for each $w \in \{x,y,z\}$. A 3-D subgrid $Q_{p,q}$ induced by a set of grid points $\{(i_x,i_y,i_z) | g_{\min}^{(x)}(p,q) \leq i_x \leq g_{\max}^{(x)}(p,q), g_{\min}^{(y)}(p,q) \leq i_y \leq g_{\max}^{(y)}(p,q), g_{\min}^{(z)}(p,q) \leq i_z \leq g_{\max}^{(z)}(p,q)\}$ is called a *center-cube* for p and q. Let $q = p + (a,b,c)$. For each $\sigma \subseteq \{x,y,z\}$, define grid points g_σ as follows:

$$\begin{aligned}
g_\emptyset &= p, \\
g_{\{x\}} &= p + (a,0,0), \\
g_{\{y\}} &= p + (0,b,0), \\
g_{\{z\}} &= p + (0,0,c), \\
g_{\{x,y\}} &= p + (a,0,c), \\
g_{\{y,z\}} &= p + (0,b,c), \\
g_{\{z,x\}} &= p + (a,0,c), \text{ and} \\
g_{\{x,y,z\}} &= p + (a,b,c) = q.
\end{aligned}$$

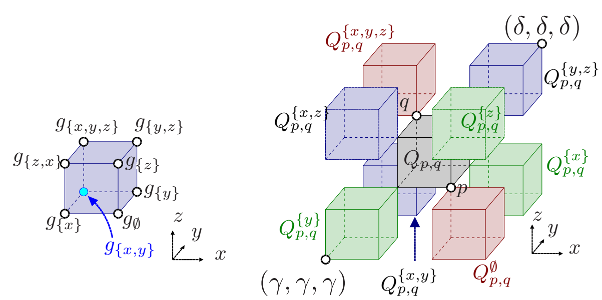
Each g_σ corresponds to a corner of $Q_{p,q}$. Figure 2 (a) shows an example of $Q_{p,q}$ and g_σ when $p_x < q_x$, $p_y > q_y$, and $p_z > q_z$. Let δ and γ be some sufficiently large and small integers, respectively, and let

$$X_\sigma = \begin{cases} \{i_x | \gamma \leq i_x \leq g_{\min}^{(x)}(p,q)\} & \text{if} \quad x \in \sigma, \\ \{i_x | g_{\max}^{(x)}(p,q) \leq i_x \leq \delta\} & \text{if} \quad x \notin \sigma, \end{cases}$$

$$Y_\sigma = \begin{cases} \{i_y | \gamma \leq i_y \leq g_{\min}^{(y)}(p,q)\} & \text{if} \quad y \in \sigma, \\ \{i_y | g_{\max}^{(y)}(p,q) \leq i_y \leq \delta\} & \text{if} \quad y \notin \sigma, \end{cases}$$

$$Z_\sigma = \begin{cases} \{i_z | \gamma \leq i_z \leq g_{\min}^{(z)}(p,q)\} & \text{if} \quad z \in \sigma, \\ \{i_z | g_{\max}^{(z)}(p,q) \leq i_z \leq \delta\} & \text{if} \quad z \notin \sigma. \end{cases}$$

For each $\sigma \subseteq \{x,y,z\}$, $Q_{p,q}^\sigma$ is a 3-D subgrid induced by a vertex set $\{(i_x,i_y,i_z) | i_x \in X_\sigma, i_y \in Y_\sigma, i_z \in Z_\sigma\}$. A 3-D grid $Q_{p,q}^\sigma$ is called a *corner-cube* for p and q. We define that $\mathcal{V}_{p,q} = V(Q_{p,q}) \cup \bigcup_{\sigma \subseteq \{x,y,z\}} V(Q_{p,q}^\sigma)$. Figure 2 (b) illustrates an example of $Q_{p,q}$ and corner cubes $Q_{p,q}^\sigma$.



(a) $Q_{p,q}$. (b) $Q_{p,q}$ and $Q_{p,q}^\sigma$.

Fig. 2. Cubes for p and q.

Let $\langle \phi, \rho \rangle$ be a 3-D embedding of a series-parallel 6-graph G with terminals s and t, and let D_s and D_t be sets of directions such that $|D_s| = d_G(s)$ and $|D_t| = d_G(t)$. Then, $\langle \phi, \rho \rangle$ is said to be *9-cubic* for $\langle D_s, D_t \rangle$ if $\phi(V(G)) \subseteq \mathcal{V}_{\phi(s),\phi(t)}$, $D_s = \{\boldsymbol{\alpha}_s^\rho(e) | e \in E_G(s)\}$, and $D_t = \{\boldsymbol{\alpha}_t^\rho(e) | e \in E_G(t)\}$, where $E_G(v)$ is the set of edges incident to v in G.

IV-B. Feasible Pair

For two vectors $\boldsymbol{a} = (a_x, a_y, a_z)$ and $\boldsymbol{b} = (b_x, b_y, b_z)$, we define that $\boldsymbol{a} * \boldsymbol{b} = (a_x b_x, a_y b_y, a_z b_z)$. We denote by $\boldsymbol{a} \cdot \boldsymbol{b}$ the inner product of $\boldsymbol{a}$ and $\boldsymbol{b}$. A vector $\boldsymbol{r} \in \{-1, 1\}^3$ is called a *diagonal direction*. For a diagonal direction $\boldsymbol{r}$, let $\mathcal{D}^+_{\boldsymbol{r}} = \{(1,0,0)*\boldsymbol{r}, (0,1,0)*\boldsymbol{r}, (0,0,1)*\boldsymbol{r}\}$ and $\mathcal{D}^-_{\boldsymbol{r}} = \mathcal{D} - \mathcal{D}^+_{\boldsymbol{r}}$. It should be noted that $\boldsymbol{d}*\boldsymbol{r} \in \mathcal{D}^+$ if and only if $\boldsymbol{d} \in \mathcal{D}^+_{\boldsymbol{r}}$ and $\boldsymbol{d} \cdot \boldsymbol{r} = 1$. Also, $\boldsymbol{d}*\boldsymbol{r} \in \mathcal{D}^-$ if and only if $\boldsymbol{d} \in \mathcal{D}^-_{\boldsymbol{r}}$ and $\boldsymbol{d} \cdot \boldsymbol{r} = -1$.

For any $D_1, D_2 \subseteq \mathcal{D}$, $\langle D_1, D_2 \rangle$ is said to be *non-admissible* if $D_1 = \{\boldsymbol{d}\}$ and $D_2 = \{-\boldsymbol{d}\}$ for some $\boldsymbol{d} \in \mathcal{D}$. Otherwise, $\langle D_1, D_2 \rangle$ is said to be *admissible*.

For $D_1, D_2 \subseteq \mathcal{D}$ and a diagonal direction $\boldsymbol{r}$, $\langle D_1, D_2 \rangle$ is said to be *inner-directed* for $\boldsymbol{r}$ if there exist directions $\boldsymbol{d}_s \in D_s \cap \mathcal{D}^+_{\boldsymbol{r}}$ and $\boldsymbol{d}_t \in D_t \cap \mathcal{D}^-_{\boldsymbol{r}}$ such that $\boldsymbol{d}_s \cdot \boldsymbol{d}_t = 0$, and $\langle D_s - \{\boldsymbol{d}_s\}, D_t - \{\boldsymbol{d}_t\} \rangle$ is admissible.

For a series-parallel 6-graph with terminals s and t, a diagonal direction $\boldsymbol{r}$, and $D_s, D_t \subseteq \mathcal{D}$, $\langle D_s, D_t \rangle$ is said to be *feasible* for G and $\boldsymbol{r}$ if all of the following conditions are satisfied:

(1) $|D_s| = d_G(s)$,
(2) $|D_t| = d_G(t)$,
(3) $\langle D_s, D_t \rangle$ is inner-directed for $\boldsymbol{r}$ if $(s,t) \in E(G)$, and
(4) $\langle D_s, D_t \rangle$ is admissible if $(s,t) \notin E(G)$.

It should be noted that if $\langle D_s, D_t \rangle$ is feasible for some G and $\boldsymbol{r} \in \{-1,1\}^3$ then $\langle D_s, D_t \rangle$ is also admissible.

It is easy to see the following.

Lemma 4: For any series-parallel 6-graph G and any diagonal direction $\boldsymbol{r} \in \{-1,1\}^3$, there exist $D_s, D_t \subseteq \mathcal{D}$ such that $\langle D_s, D_t \rangle$ is feasible for G and $\boldsymbol{r}$. ∎

IV-C. Proof

For any grid-points p and $q = p + (a,b,c)$ in the general position, a *diagonal direction* $(a/|a|, b/|b|, c/|c|)$ is denoted by $\boldsymbol{R}_{p,q}$. Now, we are ready to show the following.

Theorem 3: For a series-parallel 6-graph G with terminals s and t, a diagonal direction $\boldsymbol{r}$, and $D_s, D_t \subseteq \mathcal{D}$ such that $\langle D_s, D_t \rangle$ is feasible for G and $\boldsymbol{r}$, there exists a 9-cubic τ-embedding $\langle \phi, \rho \rangle$ of G such that $\{\boldsymbol{\alpha}^{\rho}_s(e) | e \in E_G(s)\} = D_s$, $\{\boldsymbol{\alpha}^{\rho}_t(e) | e \in E_G(t)\} = D_t$, and $\boldsymbol{R}_{\phi(s),\phi(t)} = \boldsymbol{r}$. ∎

The proof of Theorem 3 is shown in the next section. Theorem 1 follows from Theorems 2 and 3, and Lemma 4.

V. Proof of Theorem 3 (Sketch)

The theorem is proved by induction on $|E(G)|$.

If $|E(G)| = 1$, G is a graph consisting of only one edge (s,t) and so $|D_s| = |D_t| = 1$. Since $(s,t) \in E(G)$ and $\langle D_s, D_t \rangle$ is feasible for G and a diagonal direction $\boldsymbol{r}$, $\langle D_s, D_t \rangle = \langle \{\boldsymbol{d}_s\}, \{\boldsymbol{d}_t\} \rangle$ is inner-directed for $\boldsymbol{r}$. Without loss of generality we assume that $\boldsymbol{r} = (1,1,1)$, $\boldsymbol{d}_s = (1,0,0)$, and $\boldsymbol{d}_t = (0,-1,0)$. Define a 3-D embedding $\langle \phi, \rho \rangle$ of G as follows: $\phi(s) = (0,0,0)$, $\phi(t) = (1,1,1)$, and $\rho(s,t)$ is a path connecting $\phi(s)$ and $\phi(t)$, and passing through $(1,0,0)$ and $(1,0,1)$ as shown in Fig. 3. It is easy to see that $\langle \phi, \rho \rangle$ is a 9-cubic τ-embedding of G.

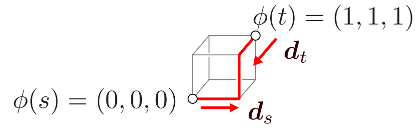


Fig. 3. 9-cubic τ-embedding of G.

Let $|E(G)| \geq 2$, and suppose that the theorem is valid for every series-parallel 6-graph with less than $|E(G)|$ edges. We distinguish two cases.

V-A. Case 1: Parallel-Composition

We consider the case when G is a parallel-composition of series-parallel graphs G_1 and G_2. We denote the terminals of G_1 and G_2 by s and t. We further distinguish two cases.

Case 1-1 $(s,t) \in E(G)$.

Without loss of generality, we assume that G_1 consists of exactly one edge (s,t) and G_2 is the graph obtained from G by deleting the edge (s,t). Then, $\langle D_s, D_t \rangle$ is inner-directed for $\boldsymbol{r}$ since $\langle D_s, D_t \rangle$ is feasible for G and $\boldsymbol{r}$. We can prove the following.

Lemma 5: There exist $\boldsymbol{d}_s \in D_s$ and $\boldsymbol{d}_t \in D_t$ such that $\langle \{\boldsymbol{d}_s\}, \{\boldsymbol{d}_t\} \rangle$ is feasible for G_1 and $\boldsymbol{r}$, and $\langle D_s - \{\boldsymbol{d}_s\}, D_t - \{\boldsymbol{d}_t\} \rangle$ is feasible for G_2 and $\boldsymbol{r}$. ∎

Thus, by the induction hypothesis, G_1 has a 9-cubic τ-embedding $\langle \phi^{(1)}, \rho^{(1)} \rangle$ for $\boldsymbol{r}$. Also, G_2 has a 9-cubic τ-embedding $\langle \phi^{(2)}, \rho^{(2)} \rangle$ for $\boldsymbol{r}$.

We can prove that we can construct a 9-cubic τ-embedding $\langle \phi, \rho \rangle$ of G for $\langle D_s, D_t \rangle$ and $\boldsymbol{r}$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$.

Case 1-2 $(s,t) \notin E(G)$.

We can prove the following.

Lemma 6: D_v can be partitioned into $D^{(1)}_v$ and $D^{(2)}_v$ for $v \in \{s,t\}$ such that $\langle D^{(1)}_s, D^{(1)}_t \rangle$ is feasible for G_1 and $\boldsymbol{r}$ and $\langle D^{(2)}_s, D^{(2)}_t \rangle$ is feasible for G_2 and $\boldsymbol{r}$. ∎

Thus, by the induction hypothesis, G_i $(i = 1,2)$ has a 9-cubic τ-embedding $\langle \phi^{(i)}, \rho^{(i)} \rangle$ for $\langle D^{(i)}_s, D^{(i)}_t \rangle$ and $\boldsymbol{r}$.

We can prove that we can construct a 9-cubic τ-embedding $\langle \phi, \rho \rangle$ of G for $\langle D_s, D_t \rangle$ and $\boldsymbol{r}$ from $\langle \phi^{(1)}, \rho^{(1)} \rangle$ and $\langle \phi^{(2)}, \rho^{(2)} \rangle$.

V-B. Case 2: Series-Composition

We consider the case when G is a series-composition of series-parallel graphs G_1 and G_2. Without loss of generality, we denote the terminals of G_1 by s and u, and those of G_2 by u and t.

Since $\langle D_s, D_t \rangle$ is admissible, there exist $\boldsymbol{d}_s \in D_s$ and $\boldsymbol{d}_t \in D_t$ satisfying $\boldsymbol{d}_s \neq -\boldsymbol{d}_t$. We further distinguish three cases.

Case 2-1 $\boldsymbol{d}_s \in \mathcal{D}^+_{\boldsymbol{r}}$ and $\boldsymbol{d}_t \in \mathcal{D}^-_{\boldsymbol{r}}$.

We can prove the following.

Lemma 7: There exist disjoint sets $D^{(s)}_u$ and $D^{(t)}_u$ of directions such that $\langle D_s, D^{(s)}_u \rangle$ is feasible for G_1 and $\boldsymbol{r}$ and $\langle D^{(t)}_u, D_t \rangle$ is feasible for G_2 and $\boldsymbol{r}$. ∎

Thus, by the induction hypothesis, G_1 has a 9-cubic τ-embedding $\langle\phi^{(1)},\rho^{(1)}\rangle$ for $\langle D_s, D_u^{(s)}\rangle$ and $\boldsymbol{r}$, and G_2 has a 9-cubic τ-embedding $\langle\phi^{(2)},\rho^{(2)}\rangle$ for $\langle D_u^{(t)}, D_t\rangle$ and $\boldsymbol{r}$.

We can prove that we can construct a 9-cubic τ-embedding $\langle\phi,\rho\rangle$ of G for $\langle D_s, D_t\rangle$ and $\boldsymbol{r}$ from $\langle\phi^{(1)},\rho^{(1)}\rangle$ and $\langle\phi^{(2)}, \rho^{(2)}\rangle$.

Case 2-2 $\boldsymbol{d}_s \in \mathcal{D}_{\boldsymbol{r}}^{+}$ and $\boldsymbol{d}_t \in \mathcal{D}_{\boldsymbol{r}}^{+}$ or $\boldsymbol{d}_s \in \mathcal{D}_{\boldsymbol{r}}^{-}$ and $\boldsymbol{d}_t \in \mathcal{D}_{\boldsymbol{r}}^{-}$.

It should be noted that $\boldsymbol{d}_s \cdot \boldsymbol{r} = \boldsymbol{d}_t \cdot \boldsymbol{r}$. Let $\boldsymbol{r}_s = \boldsymbol{r}$ if $\boldsymbol{d}_s \cdot \boldsymbol{r} = 1$ and $\boldsymbol{r}_s = -\boldsymbol{r}$ otherwise, and let $\boldsymbol{r}_u = -\boldsymbol{r}_s$. We can prove the following.

Lemma 8: There exist disjoint sets $D_u^{(s)}$ and $D_u^{(t)}$ of directions such that $\langle D_s, D_u^{(s)}\rangle$ is feasible for G_1 and $\boldsymbol{r}_s$ and $\langle D_u^{(t)}, D_t\rangle$ is feasible for G_2 and $\boldsymbol{r}_u$. ■

Thus, by the induction hypothesis, G_1 has a 9-cubic τ-embedding $\langle\phi^{(1)},\rho^{(1)}\rangle$ for $\langle D_s, D_u^{(s)}\rangle$ and $\boldsymbol{r}_s$, and G_2 has a 9-cubic τ-embedding $\langle\phi^{(2)},\rho^{(2)}\rangle$ for $\langle D_u^{(t)}, D_t\rangle$ and $\boldsymbol{r}_u$.

We can prove that we can construct a 9-cubic τ-embedding $\langle\phi,\rho\rangle$ of G for $\langle D_s, D_t\rangle$ and $\boldsymbol{r}$ from $\langle\phi^{(1)},\rho^{(1)}\rangle$ and $\langle\phi^{(2)}, \rho^{(2)}\rangle$.

Case 2-3 $\boldsymbol{d}_s \in \mathcal{D}_{\boldsymbol{r}}^{-}$ and $\boldsymbol{d}_t \in \mathcal{D}_{\boldsymbol{r}}^{+}$.

Let $\boldsymbol{r}_s = \boldsymbol{r} + 2\boldsymbol{d}_s$ and $\boldsymbol{r}_u = -\boldsymbol{r}_s$. It should be noted that $\boldsymbol{r}_s \in \{-1,1\}^3$, since $\boldsymbol{d}_s \cdot \boldsymbol{r} = -1$. We can prove the following.

Lemma 9: There exist disjoint sets $D_u^{(s)}$ and $D_u^{(t)}$ of directions such that $\langle D_s, D_u^{(s)}\rangle$ is feasible for G_1 and $\boldsymbol{r}_s$ and $\langle D_u^{(t)}, D_t\rangle$ is feasible for G_2 and $\boldsymbol{r}_u$. ■

Thus, by the induction hypothesis, G_1 has a 9-cubic τ-embedding $\langle\phi^{(1)},\rho^{(1)}\rangle$ for $\langle D_s, D_u^{(s)}\rangle$ and $\boldsymbol{r}_s$, and G_2 has a 9-cubic τ-embedding $\langle\phi^{(2)},\rho^{(2)}\rangle$ for $\langle D_u^{(t)}, D_t\rangle$ and $\boldsymbol{r}_u$.

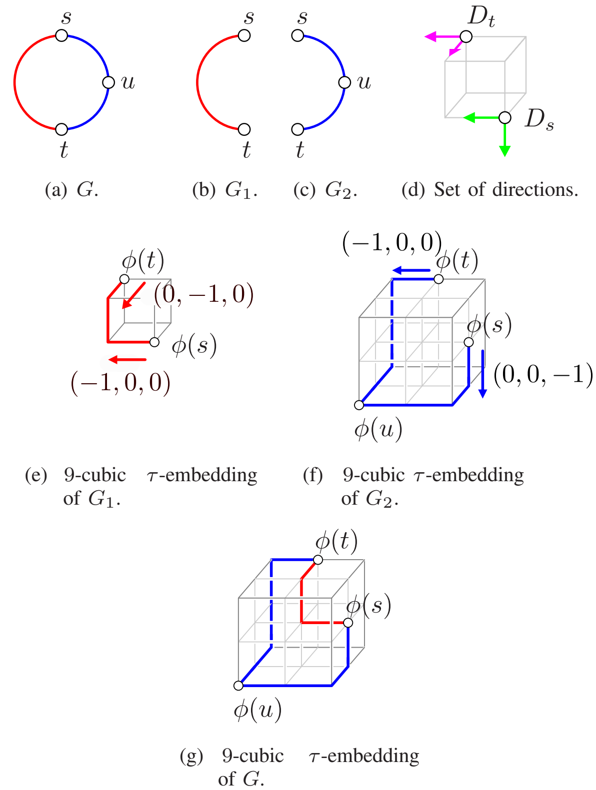
We can prove that we can construct a 9-cubic τ-embedding $\langle\phi,\rho\rangle$ of G for $\langle D_s, D_t\rangle$ and $\boldsymbol{r}$ from $\langle\phi^{(1)},\rho^{(1)}\rangle$ and $\langle\phi^{(2)}, \rho^{(2)}\rangle$.

This completes the proof of Theorem 3.

VI. Examples

A series-parallel 6-graph G shown in Fig. 4(a) is a parallel-composition of series-parallel 6-graphs G_1 and G_2 shown in Fig. 4(b) and (c), respectively. Given $D_s = \{(-1,0,0), (0,0,-1)\}$, $D_t = \{(-1,0,0),(0,-1,0)\}$, and a diagonal direction $\boldsymbol{r} = (-1,1,1)$, there exist $\boldsymbol{d}_s = (-1,0,0)$ and $\boldsymbol{d}_t = (0,-1,0)$ such that $\langle\{\boldsymbol{d}_s\},\{\boldsymbol{d}_t\}\rangle$ is feasible for G_1 and $\boldsymbol{r}$, and $\langle\{(0,0,-1)\},\{(-1,0,0)\}\rangle$ is feasible for G_2 and $\boldsymbol{r}$ by Lemma 5.

A 9-cubic τ-embedding $\langle\phi^{(1)},\rho^{(1)}\rangle$ of G_1 for $\langle\{\boldsymbol{d}_s\},\{\boldsymbol{d}_t\}\rangle$ and $\boldsymbol{r}$ is shown in Fig. 4(e), and a 9-cubic τ-embedding $\langle\phi^{(2)}, \rho^{(2)}\rangle$ of G_2 for $\langle\{(0,0,-1)\},\{(-1,0,0)\}\rangle$ and $\boldsymbol{r}$ is shown in Fig. 4(f). We obtain a 9-cubic τ-embedding $\langle\phi,\rho\rangle$ of G for $\langle D_s, D_t\rangle$ and $\boldsymbol{r}$ from $\langle\phi^{(1)},\rho^{(1)}\rangle$ and $\langle\phi^{(2)},\rho^{(2)}\rangle$ as shown in Fig. 4(g). The 9-cubic τ-embedding $\langle\phi,\rho\rangle$ of G induces a 2-bend 3-D orthogonal drawing of G by Theorem 2.



(a) G. (b) G_1. (c) G_2. (d) Set of directions.

(e) 9-cubic τ-embedding of G_1. (f) 9-cubic τ-embedding of G_2.

(g) 9-cubic τ-embedding of G.

Fig. 4. Example of parallel-composition.

References

[1] G. Battista, G. Liotta, and F. Vargiu, "Spirality and optimal orthogonal drawings," *SIAM J. Comput.*, vol. 27, pp. 1764–1811, 1998.

[2] T. Biedl and G. Kant, "A better heuristic for orthogonal graph drawings," *Computational Geometry*, vol. 9, pp. 159–180, 1998.

[3] P. Eades, C. Strik, and S. Whitesides, "The techniques of komologorov and bardzin for three-dimensional orthogonal graph drawing," *Information Processing Letters*, vol.60, pp. 97–103, 1996.

[4] P. Eades, A. Symvonis, and S. Whitesides, "Three-dimensional orthogonal graph drawing algorithms," *Discrete Applied Mathematics*, vol.103, pp. 55–87, 1996.

[5] A. Garg and R. Tamassia, "On the computational complexity of upward and rectilinear planarity testing," *SIAM J. Comput.*, vol. 31, pp. 601–625, 2001.

[6] G. Kant, "Drawing planar graphs using the canonical ordering," *Algorithmica*, vol. 16, pp. 4–32, 1996.

[7] F. T.Leighton and A. L.Rosenberg, "Three-dimensional circuit layouts," *SIAM J. Comput.*, vol.15, pp. 793–813, 1986.

[8] Y. Liu, A. Morgana, and B. Simeone, "A linear algorithm for 2-bend embeddings of planar graphs in the two-dimensional grid," *Discrete Applied Mathematics*, vol. 81, pp. 69–91, 1998.

[9] K. Nomura, S. Tayu, and S. Ueno, "On the orthogonal drawing of outerplanar grphas," *IEICE Trans. Fundamentals*, pp. 1583–1588, 2005.

[10] A. Papakostas and I. G.Tollis, "Algorithm for incremental orthogonal grpah drawing in three dimensions," *J. Graph Algorithms and Applications*, vol.3, pp. 81–115, 1999.

[11] S. T.Obenaus and T. H.Szymanski, "Embedding of star graphs into optical meshes without bends," *J. Parallel Distrib. Comput.*, vol.44, pp. 97–106, 1997.

[12] S. Tayu, K. Nomura, and S. Ueno, "On the Two-Dimensional Orthogonal Drawing of Series-Parallel Graphs," *Proc. IEEE International Symposium on Circuits and Systems ISCAS'06*, pp. 1796 –1799, 2006.

[13] J. Valdes, R. Tarjan, and E. Lawler, "The recognition of series parallel digraphs," SIAM J. Comput., vol.11, pp. 298–313, 1982.

[14] D.R. Wood, "Optoimal three-dimensional orthogonal graph drawing in the general position model," *Theoretical Computer Science*, vol.299, pp. 151–178, 2003.