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On the two-dimensional orthogonal drawing of series-parallel graphs

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ABSTRACT

It has been known that every planar 4-graph has a 2-bend 2-D orthogonal drawing, with the only exception being the octahedron, every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception being K_4, and every outerplanar 3-graph with no triangles has a 0-bend 2-D orthogonal drawing. We show in this paper that every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.

1. Introduction

We consider the problem of generating orthogonal drawings of graphs in the plane. The problem has obvious applications in the design of VLSI circuits and optoelectronic integrated systems: see for example [7,10].

Throughout this paper, we consider simple connected graphs G with vertex set $V(G)$ and edge set $E(G)$. We denote by $d_G(v)$ the degree of a vertex v in G, and by $\Delta(G)$ the maximum degree of vertices of G. G is called a *k-graph* if $\Delta(G) \leq k$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. Such a drawing of a planar graph G is called a *2-D drawing* of G. A *2-D orthogonal drawing* of a planar graph G is a 2-D drawing of G such that each edge is drawn by a sequence of contiguous horizontal and vertical line segments. Notice that a graph G has a 2-D orthogonal drawing only if $\Delta(G) \leq 4$. A 2-D orthogonal drawing with no more than b bends per edge is called a *b-bend 2-D orthogonal drawing*.

Biedl and Kant [1], and Liu, Morgana, and Simeone [5] showed that every planar 4-graph has a 2-bend 2-D orthogonal drawing, with the only exception being the octahedron shown in Fig. 1(a), which has a 3-bend 2-D orthogonal drawing, as shown in Fig. 1(b). Moreover, Kant [4] showed that every planar 3-graph has a 1-bend 2-D orthogonal drawing with the only exception being K_4 shown in Fig. 1(c), which has a 2-bend 2-D orthogonal drawing, as shown in Fig. 1(d). Zhou and Nishizeki [11] showed a linear time algorithm to generate a 1-bend 2-D orthogonal drawing for a series-parallel 3-graph. Nomura, Tayu, and Ueno [6] showed that every outerplanar 3-graph has a 0-bend 2-D orthogonal drawing if and only if it contains no triangle as a subgraph. On the other hand, Garg and Tamassia proved that it is $\mathcal{NP}$-complete to decide if a given planar 4-graph has a 0-bend 2-D orthogonal drawing [3]. Di Battista, Liotta, and Vargiu showed that the problem can be solved in polynomial time for planar 3-graphs and series-parallel graphs [2]. Rahman, Egi, and Nishizeki [8] showed that the problem can be solved in linear time for series-parallel 3-graphs.

We show in this paper the following theorem.

Theorem 1. *Every series-parallel 4-graph has a 1-bend 2-D orthogonal drawing.* □

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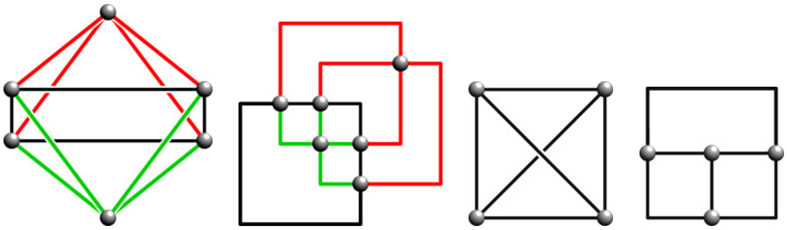
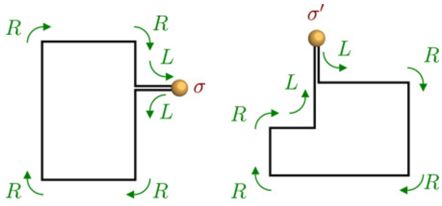


(a) Octahedron. (b) 3-bend 2-D orthogonal drawing of octahedron. (c) K_4. (d) 2-bend 2-D orthogonal drawing of K_4.

Fig. 1. Octahedron, K_4, and their 2-D orthogonal drawings.



(a) Λ_1. (b) Λ'_1.

Fig. 2. Shape-equivalent polygons Λ_1 and Λ'_1.

The proof of Theorem 1 is constructive and provides a polynomial-time algorithm to generate such a drawing for a series-parallel 4-graph.

2. Preliminaries

A *series-parallel graph* is defined recursively as follows.

(1) A graph consisting of two vertices joined by a single edge is a series-parallel graph. The vertices are the *terminals*.
(2) If G_1 is a series-parallel graph with terminals s_1 and t_1, and G_2 is a series-parallel graph with terminals s_2 and t_2, then a graph G obtained by either of the following operations is also a series-parallel graph:
  (i) *Series-composition*: identify t_1 with s_2. Vertices s_1 and t_2 are the terminals of G.
  (ii) *Parallel-composition*: identify s_1 and s_2 into a vertex s, and t_1 and t_2 into t. Vertices s and t are the terminals of G.

A series-parallel graph G is naturally associated with a binary tree $T(G)$, which is called a *decomposition tree* of G. The nodes of $T(G)$ are of three types, S-nodes, P-nodes, and Q-nodes. $T(G)$ is defined recursively as follows:

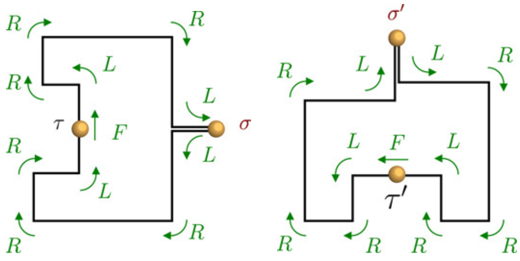
(1) If G is a single edge, then $T(G)$ consists of a single Q-node.
(2-i) If G is obtained from series-parallel graphs G_1 and G_2 by the series-composition, then the root of $T(G)$ is an S-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.
(2-ii) If G is obtained from series-parallel graphs G_1 and G_2 by the parallel-composition, then the root of $T(G)$ is a P-node, and $T(G)$ has subtrees $T(G_1)$ and $T(G_2)$ rooted at the children of the root of G.

Notice that the leaves of $T(G)$ are the Q-nodes, and an internal node of $T(G)$ is either an S-node or P-node. Notice, also, that every P-node has at most one Q-node as a child, since G is a simple graph. If G has n vertices then $T(G)$ has $\mathcal{O}(n)$ nodes, and $T(G)$ can be constructed in $\mathcal{O}(n)$ time [9]. It should be noted that the decomposition tree defined here is slightly different from the well-known SPQ-tree for a series-parallel graph.

A polygon is said to be *rectilinear* if every edge of the polygon is parallel to the horizontal or the vertical axes. Let Λ and Λ' be rectilinear polygons with distinguished vertices σ and σ', respectively. Λ and Λ' are said to be *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left and right turns for Λ and Λ'. Fig. 2 shows shape-equivalent rectilinear polygons Λ_1 and Λ'_1 whose corresponding sequence is (L, R, R, R, R, L), where L and R denote left and right turns, respectively.

Let Λ be a rectilinear polygon with distinguished vertices σ and τ, and Λ' be a rectilinear polygon with distinguished vertices σ' and τ'. Λ and Λ' are *shape-equivalent* if walking clockwise around Λ and Λ' from σ and σ', respectively, we have the same sequence of left turns, right turns, and the direction (left turn, right turn, or go straight) at τ and τ' for Λ and Λ', respectively. Fig. 3 shows shape-equivalent rectilinear polygons Λ_2 and Λ'_2 whose corresponding sequence is $S = (L, R, R, R, L, F, L, R, R, R, L)$, where F denotes the direction of going straight at τ and τ'. On the other hand, a rectilinear polygon shown in Fig. 4 is not shape-equivalent to Λ_2 or Λ'_2, since the sequence $(L, R, R, R, L, R^{\dagger}, L, R, R, R, L)$ is different from S, where $R^{\dagger}$ denotes the right turn at τ''.

Any two rectilinear rectangles with no distinguished vertex are defined to be *shape-equivalent*.



(a) Λ_2. (b) Λ'_2.

Fig. 3. Shape-equivalent polygons Λ_2 and Λ'_2 containing τ and τ', respectively.

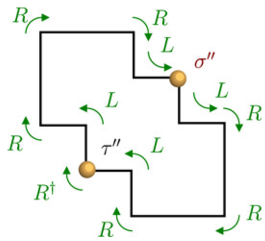
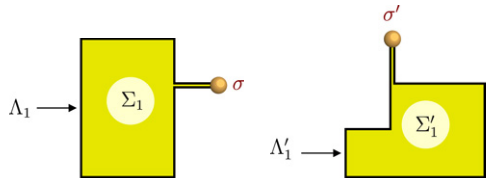
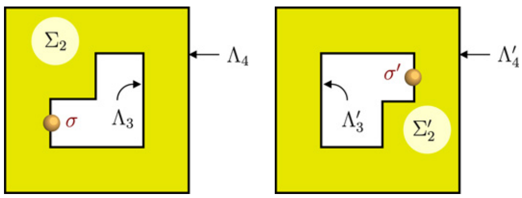


Fig. 4. A polygon not shape-equivalent to Λ_2, Λ'_2 in Fig. 3.



(a) Σ_1. (b) Σ'_1.

Fig. 5. Shape-equivalent regions Σ_1 and Σ'_1.



(a) Σ_2. (b) Σ'_2.

Fig. 6. Shape-equivalent regions bounded by two polygons Σ_2 and Σ'_2.

Let Σ and Σ' be regions bounded by rectilinear polygons Λ and Λ', respectively. Σ and Σ' are said to be *shape-equivalent* if Λ and Λ' are shape-equivalent. Regions Σ_1 and Σ'_1 shown in Fig. 5 are shape-equivalent, since bounding polygons Λ_1 and Λ'_1 are shape-equivalent, as seen in Fig. 2.

Let Λ_a and Λ_b be rectilinear polygons such that Λ_a is enclosed by Λ_b, and Σ be a region bounded by Λ_a and Λ_b. Let Λ'_a and Λ'_b be rectilinear polygons such that Λ'_a is enclosed by Λ'_b, and Σ' be a region bounded by Λ'_a and Λ'_b. Σ and Σ' are *shape-equivalent* if Λ_a and Λ'_a are shape-equivalent, and Λ_b and Λ'_b are shape-equivalent. Regions Σ_2 and Σ'_2 shown in Fig. 6 are shape-equivalent since bounding polygons Λ_3 and Λ'_3 are shape-equivalent, and Λ_4 and Λ'_4 are shape-equivalent.

A region is said to be *rectilinear* if it is bounded by rectilinear polygons.

3. Proof of Theorem 1

Let G be a series-parallel 4-graph with terminals s and t. We generate for G several 1-bend 2-D orthogonal drawings in regions of distinct shapes depending on $d_G(s)$ and $d_G(t)$. Such a region is shape-equivalent to a rectilinear region $\Pi(d_G(s), d_G(t))_i$ [$\Pi(d_G(t), d_G(s))_i$] shown in Fig. 7 for some integer i, if $d_G(s) \le d_G(t)$ [$d_G(t) \le d_G(s)$]. A region $\Pi(d_1, d_2)_i$ is also referred to as $\Pi(d_2, d_1)_i$. The number $\nu(d_G(s), d_G(t))$ of distinct shapes is no more than 4 for every pair of $d_G(s)$ and

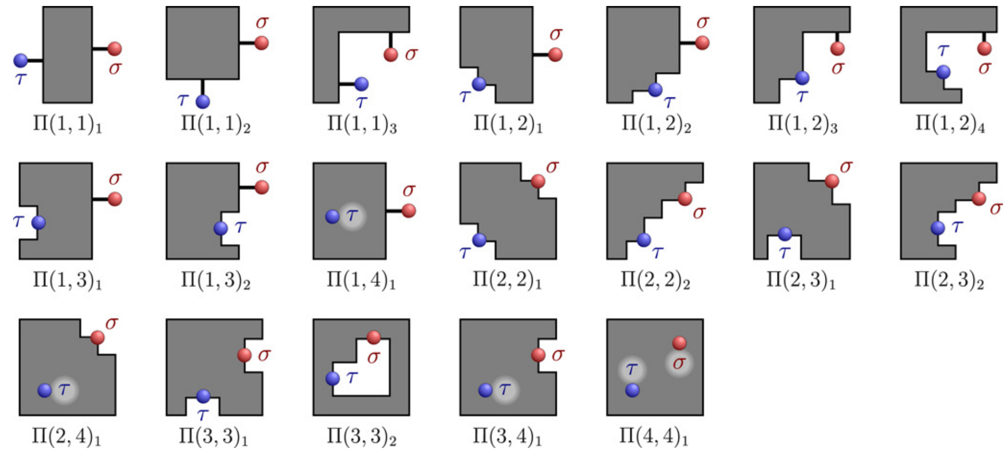
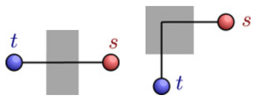


Fig. 7. Regions.



(a) N_1-drawing. (b) N_2-drawing.

Fig. 8. N-drawings of an edge.

$d_G(t)$. More precisely, $\nu(d_G(s), d_G(t))$, $d_G(s) \le d_G(t)$, is: 3 if $d_G(s) = d_G(t) = 1$; 4 if $d_G(s) = 1$ and $d_G(t) = 2$; 2 if $d_G(s) = 1$ and $d_G(t) = 3$; 1 if $d_G(s) = 1$ and $d_G(t) = 4$; 2 if $d_G(s) = d_G(t) = 2$; 2 if $d_G(s) = 2$ and $d_G(t) = 3$; 1 if $d_G(s) = 2$ and $d_G(t) = 4$; 2 if $d_G(s) = d_G(t) = 3$; 1 if $d_G(s) = 3$ and $d_G(t) = 4$; 1 if $d_G(s) = d_G(t) = 4$, as seen in Fig. 7.

Let Σ be a rectilinear region with distinguished vertices σ and τ. A 1-bend 2-D orthogonal drawing of G in Σ is called an *N-drawing* of G generated in Σ if s is mapped to one of σ and τ, and t is mapped to the other one. We will show that G has an N-drawing generated in a region shape-equivalent to $\Pi(d_G(s), d_G(t))_i$ [$\Pi(d_G(t), d_G(s))_i$] for each i, $1 \le i \le \nu(d_G(s), d_G(t))$ [$1 \le i \le \nu(d_G(t), d_G(s))$]. We have two exceptions. If $d_G(s) = d_G(t) = 1$ and $(s, t) \in E(G)$ then we show that G has an N-drawing generated in a region shape-equivalent to $\Pi(1, 1)_i$ for each i, $1 \le i \le 2$. Also, if $d_G(s) = d_G(t) = 3$ and $(s, t) \in E(G)$ then we show that G has an N-drawing generated in a region shape-equivalent to $\Pi(3, 3)_1$.

It is sufficient to prove the following theorem.

Theorem 2. *Every series-parallel 4-graph with terminals s and t has an N-drawing generated in a region shape-equivalent to $\Pi(d_G(s), d_G(t))_i$ for $1 \le i \le \nu(d_G(s), d_G(t))$ with the exception that $1 \le i \le 2$ if $d_G(s) = d_G(t) = 1$ and $(s, t) \in E(G)$, and that $i = 1$ if $d_G(s) = d_G(t) = 3$ and $(s, t) \in E(G)$.*

Proof. The theorem is proved by induction on $|E(G)|$. An N-drawing of G in a region shape-equivalent to $\Pi(d_G(s), d_G(t))_i$ is called an *N_i-drawing* of G.

If $|E(G)| = 1$, G is a graph consisting of just an edge (s, t). Such a graph has an N_1-drawing and N_2-drawing, as shown in Fig. 8.

Assume that $|E(G)| \ge 2$, and Theorem 2 holds for any series-parallel 4-graph with at most $|E(G)| - 1$ edges. We assume, without loss of generality, that $d_G(s) \le d_G(t)$. We distinguish two cases.

Case A: G is a series-composition of G_1 and G_2.

Lemma 3. *For any i, $1 \le i \le \nu(d_G(s), d_G(t))$, there exist j and k, $1 \le j \le \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $1 \le k \le \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that an N_i-drawing of G can be generated by combining an N_j-drawing of G_1 and N_k-drawing of G_2.*

Proof of Lemma 3. Table 1 shows such a pair of j and k for each i and the degrees $d_G(s)$, $d_G(t)$, and $d_{G_1}(t_1)$. Since neither $\Pi(1, 1)_3$ nor $\Pi(3, 3)_2$ appears in the columns of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ of Table 1, G_1 has every N_j-drawing indicated in the column of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ of the table, and G_2 has every N_k-drawing indicated in the column of $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ of the table, by induction hypothesis. We can see that an N_i-drawing of G can be generated by combining an N_j-drawing Γ_1 of G_1 and N_k-drawing Γ_2 of G_2 as shown in Figs. 9–15. For example, Fig. 9(a) shows that if $d_G(s) = d_G(t) = d_{G_1}(t_1) = d_{G_2}(s_2) = 1$ then an N_1-drawing of G can be generated from an N_1-drawing Γ_1 of G_1 and N_1-drawing Γ_2 of G_2 by identifying t_1 with s_2. Fig. 9(b) shows that if $d_G(s) = d_G(t) = d_{G_1}(t_1) = 1$ and $d_{G_2}(s_2) = 2$ then an N_1-drawing of G can be generated from an N_1-drawing Γ_1 of G_1 and N_1-drawing Γ_2 of G_2 by scaling Γ_1 and rotating Γ_2 appropriately, and identifying t_1 with s_2. Fig. 10(o) shows that if $d_G(s) = 1$, $d_G(t) = 2$, $d_{G_1}(t_1) = 1$, and $d_{G_2}(s_2) = 3$ then

Table 1
Pair of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ for $\Pi(d_G(s), d_G(t))_i$ when G is a series-composition.

$\Pi(d_G(s), d_G(t))_i$	$\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$	$\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$
$\Pi(1, 1)_1$	$\Pi(1, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 1)_1$
	$\Pi(1, 1)_2$	$\Pi(d_{G_2}(s_2), 1)_1$
$\Pi(1, 1)_2$	$\Pi(1, 2)_2$	$\Pi(d_{G_2}(s_2), 1)_1$
	$\Pi(1, 3)_1$	$\Pi(1, 1)_2$
$\Pi(1, 1)_3$	$\Pi(1, d_{G_1}(t_1))_2$	$\Pi(d_{G_2}(s_2), 1)_2$
$\Pi(1, 2)_1$	$\Pi(1, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 2)_1$
$\Pi(1, 2)_2$	$\Pi(1, d_{G_1}(t_1))_2$	$\Pi(d_{G_2}(s_2), 2)_1$
	$\Pi(1, 1)_1$	$\Pi(d_{G_2}(s_2), 2)_2$
$\Pi(1, 2)_3$	$\Pi(1, 2)_1$	$\Pi(d_{G_2}(s_2), 2)_2$
	$\Pi(1, 3)_2$	$\Pi(1, 2)_1$
$\Pi(1, 2)_4$	$\Pi(1, d_{G_1}(t_1))_2$	$\Pi(d_{G_2}(s_2), 2)_2$
	$\Pi(1, 1)_2$	$\Pi(d_{G_2}(s_2), 3)_1$
$\Pi(1, 3)_1$	$\Pi(1, 2)_1$	$\Pi(d_{G_2}(s_2), 3)_1$
	$\Pi(1, 3)_1$	$\Pi(1, 3)_1$
$\Pi(1, 3)_2$	$\Pi(1, d_{G_1}(t_1))_2$	$\Pi(d_{G_2}(s_2), 3)_1$
$\Pi(2, 2)_1$	$\Pi(2, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 2)_1$
$\Pi(2, 2)_2$	$\Pi(2, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 2)_2$
$\Pi(2, 3)_1$	$\Pi(2, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 3)_1$
$\Pi(2, 3)_2$	$\Pi(2, d_{G_1}(t_1))_2$	$\Pi(d_{G_2}(s_2), 3)_1$
$\Pi(3, 3)_1$	$\Pi(3, d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 3)_1$
	$\Pi(3, 1)_2$	$\Pi(d_{G_2}(s_2), 3)_1$
$\Pi(3, 3)_2$	$\Pi(3, 2)_2$	$\Pi(d_{G_2}(s_2), 3)_1$
	$\Pi(3, 3)_1$	$\Pi(1, 3)_2$
$\Pi(d_G(s), 4)_1$	$\Pi(d_G(s), d_{G_1}(t_1))_1$	$\Pi(d_{G_2}(s_2), 4)_1$

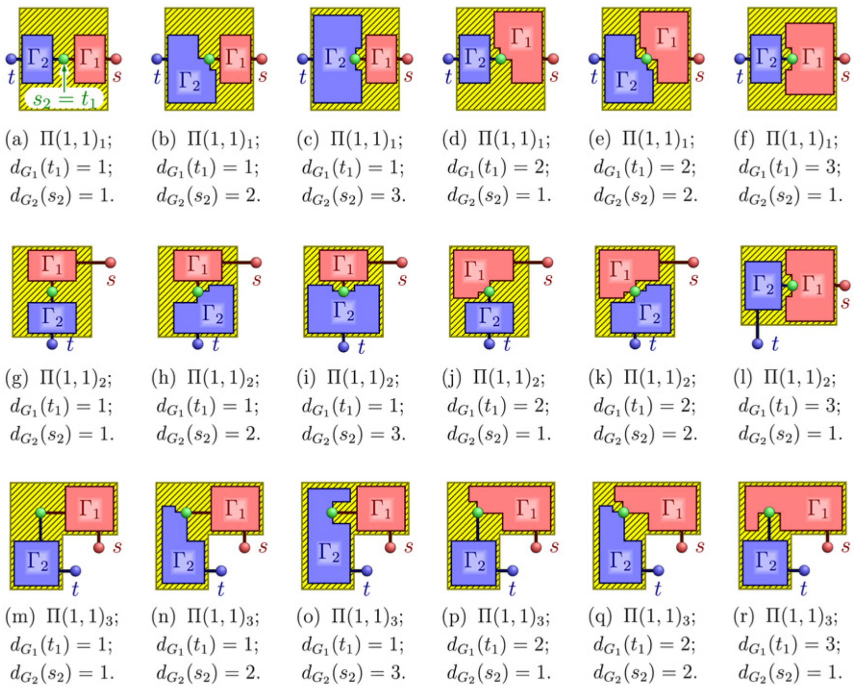


Fig. 9. N-drawings of G when G is a series-composition and $d_G(s) = d_G(t) = 1$.

an N_3-drawing of G can be generated from an N_1-drawing Γ_1 of G_1 and N_2-drawing Γ_2 of G_2 by scaling and rotating Γ_1 and flipped Γ_2 appropriately, and identifying t_1 with s_2. □

Case B: G is a parallel-composition of G_1 and G_2.

We assume, without loss of generality, that if G has edge (s, t) then G_1 consists of exactly one edge (s_1, t_1).

Lemma 4. *For any i, $1 \le i \le \nu(d_G(s), d_G(t))$, there exist j and k, $1 \le j \le \nu(d_{G_1}(s_1), d_{G_1}(t_1))$ and $1 \le k \le \nu(d_{G_2}(s_2), d_{G_2}(t_2))$, such that an N_i-drawing of G can be generated by combining an N_j-drawing of G_1 and N_k-drawing of G_2.*

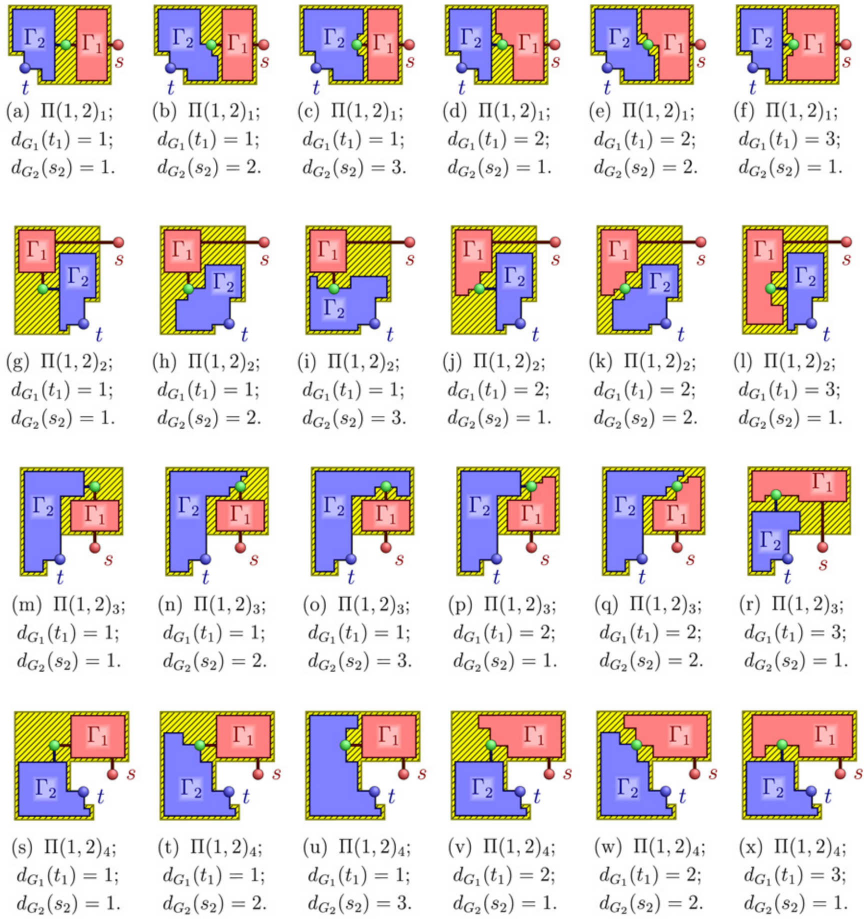


Fig. 10. *N*-drawings of *G* when *G* is a series-composition, $d_G(s) = 1$, and $d_G(t) = 2$.

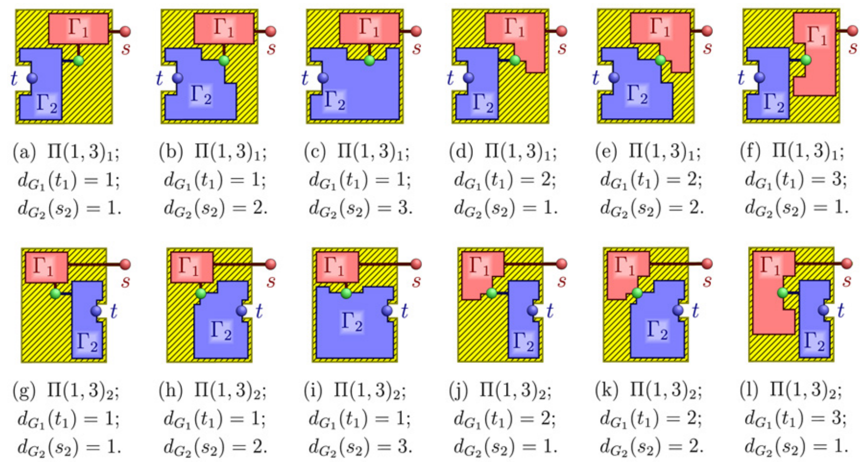
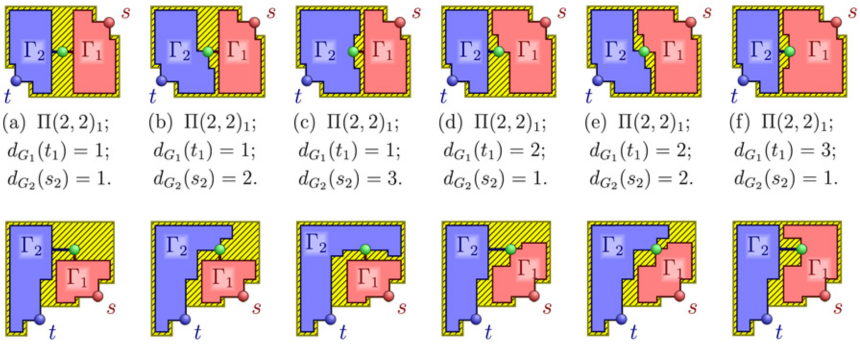


Fig. 11. *N*-drawings of *G* when *G* is a series-composition, $d_G(s) = 1$, and $d_G(t) = 3$.

Proof of Lemma 4. We assume, without loss of generality, that $d_{G_1}(s_1) + d_{G_2}(s_2) \le d_{G_1}(t_1) + d_{G_2}(t_2)$, and $d_{G_1}(s_1) = \min\{d_{G_1}(s_1), d_{G_1}(t_1), d_{G_2}(s_2), d_{G_2}(t_2)\}$. We distinguish two cases:

Case B-1: $(s, t) \notin E(G)$.

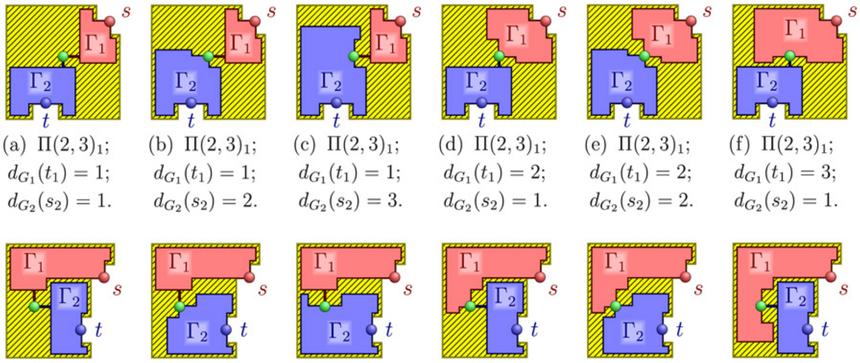
Table 2 shows such a pair of *j* and *k* for each *i* and the degrees $d_G(s)$, $d_G(t)$, $d_{G_1}(s_1)$, and $d_{G_1}(t_1)$.



(a) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (b) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (c) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (d) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (e) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (f) $\Pi(2,2)_1$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

(g) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (h) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (i) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (j) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (k) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (l) $\Pi(2,2)_2$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

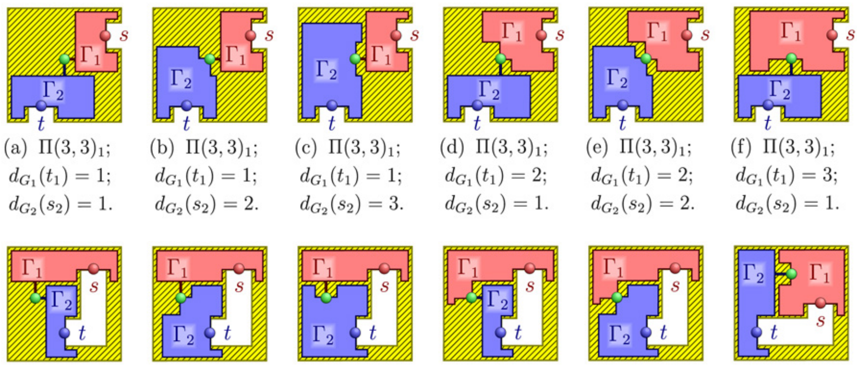
Fig. 12. *N*-drawings of *G* when *G* is a series-composition and $d_G(s) = d_G(t) = 2$.



(a) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (b) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (c) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (d) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (e) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (f) $\Pi(2,3)_1$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

(g) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (h) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (i) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (j) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (k) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (l) $\Pi(2,3)_2$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

Fig. 13. *N*-drawings of *G* when *G* is a series-composition, $d_G(s) = 2$, and $d_G(t) = 3$.



(a) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (b) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (c) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (d) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (e) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (f) $\Pi(3,3)_1$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

(g) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 1$. (h) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 2$. (i) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 1$; $d_{G_2}(s_2) = 3$. (j) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 1$. (k) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 2$; $d_{G_2}(s_2) = 2$. (l) $\Pi(3,3)_2$; $d_{G_1}(t_1) = 3$; $d_{G_2}(s_2) = 1$.

Fig. 14. *N*-drawings of *G* when *G* is a series-composition and $d_G(s) = d_G(t) = 3$.

Since $(s_1, t_1) \notin E(G_1)$ and $(s_2, t_2) \notin E(G_2)$, G_1 and G_2 have every N_j-drawing and N_k-drawing indicated in the columns of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ of Table 2, respectively, by induction hypothesis. We can see that an N_i-drawing of G can be generated by combining an N_j-drawing Γ_1 of G_1 and N_k-drawing Γ_2 of G_2 as shown in Fig. 16.

Fig. 15. N-drawings of G when G is a series-composition and $d_G(t) = 4$.

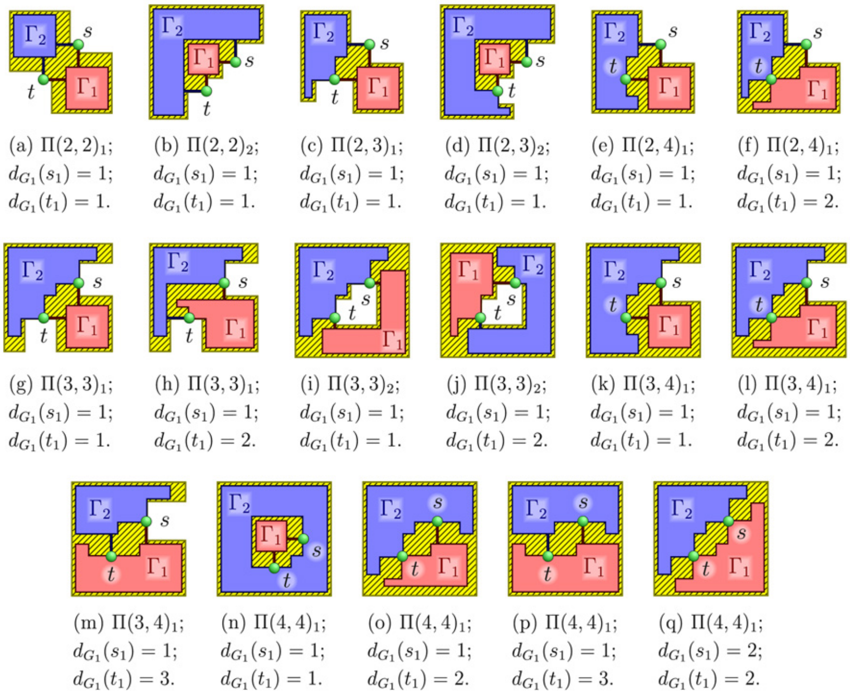
Table 2
Pair of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ for $\Pi(d_G(s), d_G(t))_i$ when G is a parallel-composition and $(s, t) \notin E(G)$.

$\Pi(d_G(s), d_G(t))_i$	$\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$	$\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$
$\Pi(2,2)_1$	$\Pi(1,1)_2$	$\Pi(1,1)_2$
$\Pi(2,2)_2$	$\Pi(1,1)_2$	$\Pi(1,1)_3$
$\Pi(2,3)_1$	$\Pi(1,1)_2$	$\Pi(1,2)_2$
$\Pi(2,3)_2$	$\Pi(1,1)_2$	$\Pi(1,2)_4$
$\Pi(2,4)_1$	$\Pi(1,1)_2$	$\Pi(1,3)_2$
	$\Pi(1,2)_2$	$\Pi(1,2)_2$
$\Pi(3,3)_1$	$\Pi(1,1)_2$	$\Pi(2,2)_2$
	$\Pi(1,2)_1$	$\Pi(2,1)_3$
$\Pi(3,3)_2$	$\Pi(1,1)_3$	$\Pi(2,2)_2$
	$\Pi(1,2)_2$	$\Pi(2,1)_4$
$\Pi(3,4)_1$	$\Pi(1,1)_2$	$\Pi(2,3)_2$
	$\Pi(1,2)_2$	$\Pi(2,2)_2$
	$\Pi(1,3)_2$	$\Pi(2,1)_2$
$\Pi(4,4)_1$	$\Pi(1,1)_2$	$\Pi(3,3)_2$
	$\Pi(1,2)_1$	$\Pi(3,2)_2$
	$\Pi(1,3)_1$	$\Pi(3,1)_1$
	$\Pi(2,2)_2$	$\Pi(2,2)_2$

Case B-2: $(s, t) \in E(G)$.

Notice that $d_{G_1}(s_1) = d_{G_1}(t_1) = 1$, and $(s_2, t_2) \notin E(G_2)$, since G_1 consists of exactly one edge (s_1, t_1) by the assumption of Case B. Notice, also, that $\Pi(d_G(s), d_G(t))_i \neq \Pi(3,3)_2$ by the exception of Theorem 2. Table 3 shows such a pair of j and k for each i and the degrees $d_G(s)$, $d_G(t)$, $d_{G_1}(s_1)$, and $d_{G_1}(t_1)$. It is easy to see that G_1 and G_2 have every N_j-drawing and N_k-drawing indicated in the columns of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ of Table 3, respectively, by induction hypothesis. We can see that an N_i-drawing of G can be generated by combining an N_j-drawing Γ_1 of G_1 and N_k-drawing Γ_2 of G_2 as shown in Fig. 16. □

From Lemmas 3 and 4, and the induction hypothesis, we obtain Theorem 2. □



(a) $\Pi(2,2)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(b) $\Pi(2,2)_2$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(c) $\Pi(2,3)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(d) $\Pi(2,3)_2$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(e) $\Pi(2,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(f) $\Pi(2,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=2$.
(g) $\Pi(3,3)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(h) $\Pi(3,3)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=2$.
(i) $\Pi(3,3)_2$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(j) $\Pi(3,3)_2$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=2$.
(k) $\Pi(3,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(l) $\Pi(3,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=2$.
(m) $\Pi(3,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=3$.
(n) $\Pi(4,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=1$.
(o) $\Pi(4,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=2$.
(p) $\Pi(4,4)_1$; $d_{G_1}(s_1)=1$; $d_{G_1}(t_1)=3$.
(q) $\Pi(4,4)_1$; $d_{G_1}(s_1)=2$; $d_{G_1}(t_1)=2$.

Fig. 16. *N*-drawings of *G* when *G* is a parallel-composition.

Table 3
Pair of $\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$ and $\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$ for $\Pi(d_G(s), d_G(t))_i$ when G is a parallel-composition and $(s,t) \in E(G)$.

$\Pi(d_G(s), d_G(t))_i$	$\Pi(d_{G_1}(s_1), d_{G_1}(t_1))_j$	$\Pi(d_{G_2}(s_2), d_{G_2}(t_2))_k$
$\Pi(2,2)_1$	$\Pi(1,1)_2$	$\Pi(1,1)_2$
$\Pi(2,2)_2$	$\Pi(1,1)_2$	$\Pi(1,1)_3$
$\Pi(2,3)_1$	$\Pi(1,1)_2$	$\Pi(1,2)_2$
$\Pi(2,3)_2$	$\Pi(1,1)_2$	$\Pi(1,2)_4$
$\Pi(2,4)_1$	$\Pi(1,1)_2$	$\Pi(1,3)_2$
$\Pi(3,3)_1$	$\Pi(1,1)_2$	$\Pi(2,2)_2$
$\Pi(3,4)_1$	$\Pi(1,1)_2$	$\Pi(2,3)_2$
$\Pi(4,4)_1$	$\Pi(1,1)_2$	$\Pi(3,3)_2$

4. Algorithm

The proof of Theorem 2 in the previous section provides a recursive algorithm, 2D_DRAW shown in Fig. 17, to generate an N-drawing for a series-parallel 4-graph.

Theorem 5. *Given a series-parallel* 4*-graph G with terminals s and t, and an integer i with $1 \le i \le \nu(d_G(s), d_G(t))$, 2D_DRAW generates an N_i-drawing of G in $O(|E(G)|^2)$ time.*

Proof. The correctness of 2D_DRAW follows from the proof of Theorem 2.

We will show the time complexity of 2D_DRAW. It is not difficult to see that scaling, rotation, and reflection of $\Gamma_j(G_1)$ and $\Gamma_k(G_2)$ can be performed in $O(|E(G_1)| + |E(G_2)|) = O(|E(G)|)$ time. Thus, if $\mathcal{T}'(G)$ is the computation time of 2D_DRAW for G, we have the following recurrence relation:

$$\mathcal{T}'(G) \le \mathcal{T}'(G_1) + \mathcal{T}'(G_2) + c_1|E(G)| + c_2 \quad (1)$$

for some positive constants c_1 and c_2. Let $\mathcal{T}(m) = \max_{|E(G)|=m} \mathcal{T}'(G)$. It follows from (1) that:

$$\mathcal{T}(m) \le \max_{1 \le p \le m-1} (\mathcal{T}(p) + \mathcal{T}(m-p)) + c_1 m + c_2, \quad (2)$$

since $|E(G)| = |E(G_1)| + |E(G_2)|$ and $1 \le |E(G_1)| \le |E(G)| - 1$.

Claim 6. $\mathcal{T}(m) \le cm^2$ *for a constant* $c = c_1 + c_2 + \mathcal{T}(1)$.

```
Input:   a series-parallel 4-graph G, terminals s, t ∈ V(G),
         and an integer i with 1 ≤ i ≤ ν(d_G(s), d_G(t)).
Output:  N_i-drawing Γ_i(G) of G.
begin
  compute T(G);
  if the root of T(G) is a Q-node then
    let Γ_i(G) be an N_i-drawing;
  endif
  if the root of T(G) is an S-node then
    compute G_1 and G_2 such that G is a series composition
    of G_1 and G_2;
    choose j and k according to d_G(s), d_G(t), i, and d_{G_1}(t_1)
    as shown in Table 1;
    Γ_j(G_1) := 2D_DRAW(G_1, s_1, t_1, j);
    Γ_k(G_2) := 2D_DRAW(G_2, s_2, t_2, k);
    apply appropriate scaling, rotation, and reflection to
    Γ_j(G_1) and Γ_k(G_2), and combine them into Γ_i(G);
  endif
  if the root of T(G) is a P-node then
    if G contains edge (s, t) then
      let G_1 be the graph consisting of only one edge (s, t)
      and G_2 = G − G_1;
      choose j and k according to degrees d_G(s), d_G(t), d_{G_1}(s_1),
      d_{G_1}(t_1), d_{G_2}(s_2), and d_{G_2}(t_2) as shown in Table 2;
    else
      compute G_1 and G_2 such that G is a parallel
      composition of G_1 and G_2;
      choose j and k according to degrees d_G(s), d_G(t), d_{G_1}(s_1),
      d_{G_1}(t_1), d_{G_2}(s_2), and d_{G_2}(t_2) as shown in Table 3;
    endif
    Γ_j(G_1) := 2D_DRAW(G_1, s_1, t_1, j);
    Γ_k(G_2) := 2D_DRAW(G_2, s_2, t_2, k);
    apply appropriate scaling, rotation, and reflection to
    Γ_j(G_1) and Γ_k(G_2), and combine them into Γ_i(G);
  endif
  return Γ_i(G);
end
```

Fig. 17. Algorithm: 2D_DRAW (G, s, t, i).

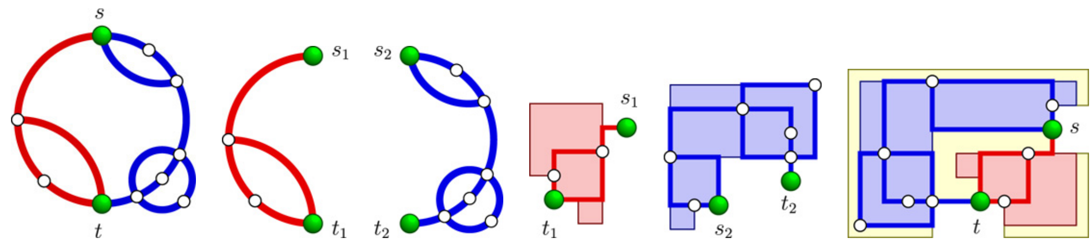
Proof of Claim 6. Claim 6 holds when $m = 1$ since $c \cdot 1^2 \geq \mathcal{T}(1)$. Assume that $m \geq 2$ and Claim 6 holds for any positive integer less than m. Then by (2), we have

$$\begin{aligned}\mathcal{T}(m) &\leq \max_{1\leq p\leq m-1} (\mathcal{T}(p) + \mathcal{T}(m-p)) + c_1 m + c_2\\ &\leq \max_{1\leq p\leq m-1} \left(cp^2 + c(m-p)^2\right) + c_1 m + c_2\\ &= cm^2 - 2\,cm + 2c + c_1 m + c_2\\ &= cm^2 - (cm - 2c) - (cm - c_1 m - c_2)\\ &\leq cm^2,\end{aligned}$$

since $m \geq 2$. □

From Claim 6, we have $\mathcal{T}(|E(G)|) = O(|E(G)|^2)$. This completes the proof of Theorem 5. □

As an example, we show an induction step to generate an N_1-drawing of a series-parallel 4-graph G shown in Fig. 18(a). G is a parallel-composition of series-parallel graphs G_1 and G_2 shown in Fig. 18(b) and (c), respectively. Since $d_G(s) = d_G(t) = 3$, we need an N_1-drawing Γ_1 of G_1 and N_3-drawing Γ_2 of G_2 by Table 2. Γ_1 and Γ_2 are shown in Fig. 18(d) and (e), respectively. Finally, an N_1-drawing Γ of G can be generated by flipping, rotating, and scaling Γ_1 and Γ_2 appropriately, and identifying s_1 with s_2, and t_1 with t_2 as shown in Fig. 18(f).



(a) G. (b) G_1. (c) G_2. (d) N_1-drawing Γ_1 of G_1. (e) N_3-drawing Γ_2 of G_2. (f) N_1-drawing of G.

Fig. 18. Example of a recursive step of algorithm.

5. Concluding remarks

We can prove that every series-parallel 6-graph has a 2-bend 3-D orthogonal drawing, which will appear in a forthcoming paper.

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