PAPER

On the Three-Dimensional Channel Routing

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SUMMARY The 3-D channel routing is a fundamental problem on the physical design of 3-D integrated circuits. The 3-D channel is a 3-D grid G and the terminals are vertices of G located in the top and bottom layers. A net is a set of terminals to be connected. The objective of the 3-D channel routing problem is to connect the terminals in each net with a Steiner tree (wire) in G using as few layers as possible and as short wires as possible in such a way that wires for distinct nets are disjoint. This paper shows that the problem is intractable. We also show that a sparse set of ν 2-terminal nets can be routed in a 3-D channel with $O(\sqrt{\nu})$ layers using wires of length $O(\sqrt{\nu})$.

key words: *3-D channel, NP-complete, routing algorithm, Steiner tree*

1. Introduction

The three-dimensional (3-D) integration is an emerging technology to implement large circuits, and currently being extensively investigated. (See [2]–[4], [7], [9], [14], [16], [19], [22] for example.) In this paper, we consider a problem on the physical design of 3-D integrated circuits.

The 3-D channel routing is a fundamental problem on the physical design of 3-D integrated circuits. The 3-D channel is a 3-D grid G consisting of *columns*, *rows*, and *layers* which are rectilinear grid planes defined by fixing x-, y-, and z-coordinates at integers, respectively. The numbers of columns, rows, and layers are called the *width*, *depth*, and *height* of G, respectively. (See Fig. 1.) G is called a (W, D, H)-*channel* if the width is W, depth is D, and height is H. A vertex of G is a grid point with integer coordinates. We assume without loss of generality that the vertex set of a (W, D, H)-channel is $\{(x, y, z) \mid x \in [W], y \in [D], z \in [H]\}$, where $[i] = \{1, 2, \ldots, i\}$ for a positive integer i. Layers defined by $z = H$ and $z = 1$ are called the *top* and *bottom layers*, respectively.

A *terminal* is a vertex of G located on the top or bottom layer. A *net* is a set of terminals to be connected. A net containing k terminals is called a k-*net*. The object of the 3-D channel routing problem is to connect the terminals in each net with a Steiner tree (wire) in G using as few layers as possible and as short wires as possible in such a way that

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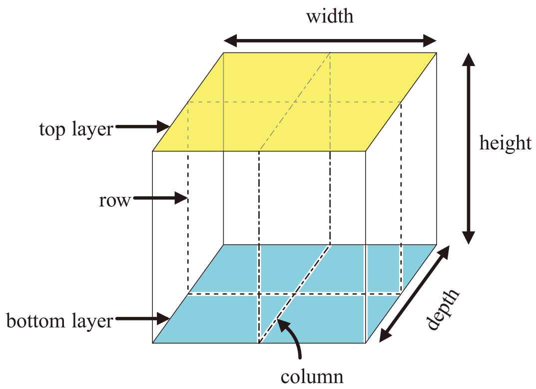


Fig. 1 3-D channel.

Steiner trees spanning distinct nets are vertex-disjoint. A set of nets is said to be *routable* in G if G has vertex-disjoint Steiner trees spanning the nets.

We first show in Sect. 2 that the 3-D channel routing problem is intractable. We next show in Sect. 3 that if G is a $(2n, 2n, 3n+1)$-channel, the terminals are located on vertices with odd x- and y-coordinates, and each net has terminals both on the top and bottom layers, then any set of n^2 2-nets is routable in G. We finally show in Sect. 4 some lower bounds for the height of a 3-D channel routing for 2-nets. In particular, we show that there exists a set of n^2 such 2-nets that cannot be routed in a $(2n, 2n, n/2 - 1)$-channel.

2. Intractability

We consider in this section the complexity of the following decision problem associated with the 3-D channel routing problem.

3-D CHANNEL ROUTING

INSTANCE: Positive integers W, D, H, a set of terminals $T \subseteq \{(x, y, z) \mid x \in [W], y \in [D], z \in \{1, H\}\}$ and a partition of T into nets $N_1, N_2, \ldots, N_\nu$.

QUESTION: Is a set of nets $\{N_1, N_2, \ldots, N_\nu\}$ routable in a (W, D, H)-channel?

We have two well-known problems as subproblems of 3-D CHANNEL ROUTING, namely, ONE-ROW CHANNEL ROUTING and TWO-ROW CHANNEL ROUTING. These problems can be stated as follows.

ONE-ROW CHANNEL ROUTING

INSTANCE: Positive integers W, H, a set of terminals

$T \subseteq \{(x, 1, z) \mid x \in [W], z \in \{1, H\}\}$ and a partition of T into nets $N_1, N_2, \ldots, N_v$.

QUESTION: Is a set of nets $\{N_1, N_2, \ldots, N_v\}$ routable in a $(W, 1, H)$-channel?

TWO-ROW CHANNEL ROUTING

INSTANCE: Positive integers W, H, a set of terminals $T \subseteq \{(x, 1, z) \mid x \in [W], z \in \{1, H\}\}$ and a partition of T into nets $N_1, N_2, \ldots, N_v$.

QUESTION: Is a set of nets $\{N_1, N_2, \ldots, N_v\}$ routable in a $(W, 2, H)$-channel?

It should be noted that TWO-ROW CHANNEL ROUTING has been known as "UNRESTRICTED" TWO-LAYER CHANNEL ROUTING in the literature. The complexity of TWO-ROW CHANNEL ROUTING is a longstanding open question posed by Johnson [10], while ONE-ROW CHANNEL ROUTING can be solved in polynomial time as shown by Dolev, Karplus, Siegel, Strong, and Ullman [8].

The purpose of this section is to show the following.

Theorem 1: 3-D CHANNEL ROUTING is NP-hard even for 2-nets. □

The complexity of TWO-ROW CHANNEL ROUTING is still open. Moreover, the complexity of the following problem is open for any fixed integer $d \geq 2$.

2.5-D CHANNEL ROUTING

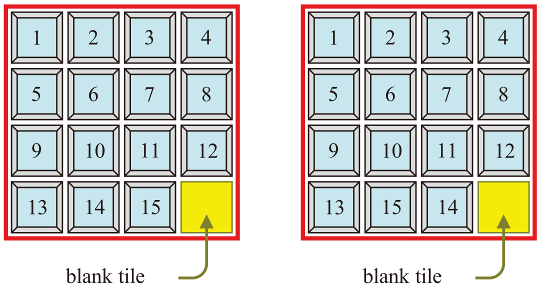
INSTANCE: Positive integers W, H, a set of terminals $T \subseteq \{(x, y, z) \mid x \in [W], y \in [d], z \in \{1, H\}\}$ and a partition of T into nets $N_1, N_2, \ldots, N_v$.

QUESTION: Is a set of nets $\{N_1, N_2, \ldots, N_v\}$ routable in a (W, d, H)-channel?

The 3-D channel routing for 2-nets is closely related to the $(n^2 - 1)$-puzzle defined below.

2.1 $(n^2 - 1)$-Puzzle

The $(n^2 - 1)$-puzzle is a generalization of the well-known 15-puzzle [12]. The $(n^2 - 1)$-*puzzle* is played on an $n \times n$ board, $n \geq 2$. There are n^2 distinct tiles on the board: one *blank tile* and $n^2 - 1$ tiles numbered from 1 to $n^2 - 1$. Each of the n^2 square locations of the board is occupied by exactly one tile. An instance of $(n^2 - 1)$-puzzle consists of two board configurations C (the *initial configuration*) and C' (the *final configuration*). A *move* is an exchange of the blank tile with a nonblank tile located on a horizontally or vertically adjacent location. The goal of the puzzle is to find a sequence of moves that transforms C to C'. The configuration C' is said to be *reachable* from C if there exists such a sequence of moves. Notice that C' is reachable from C if and only if C is reachable from C'. The configurations C and C' are said to be *reachable* with h moves if there exists a sequence of at most h moves that transforms C to C'. Figure 2 shows two unreachable configurations of 15-puzzle. This is the original 15-puzzle of Loyd [12]. Our problem is to find a shortest sequence of moves that transforms C to C' if C and C' are reachable. The corresponding decision problem is described as follows.



(a) Initial confi guration. (b) Final confi guration.

Fig. 2 Unreachable configurations of 15-puzzle.

$(n^2 - 1)$-PUZZLE

INSTANCE: Two n^2 board configurations C and C', and a positive integer h.

QUESTION: Are C and C' reachable with h moves?

Ratner and Warmuth [15] showed the following.

Theorem I $(n^2 - 1)$-PUZZLE is NP-complete. □

2.2 Proof of Theorem 1

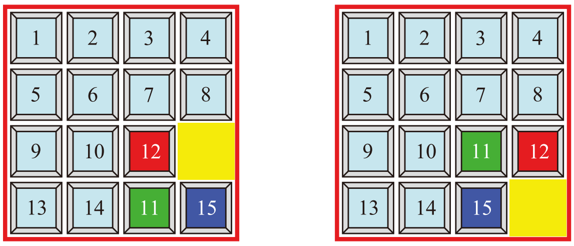
We reduce (n^2-1)-PUZZLE to 3-D CHANNEL ROUTING. The $(n^2 - 1)$-puzzle is naturally associated with a 3-D channel routing for 2-nets as follows. The configurations C and C' are corresponding to the top and bottom layers. A terminal is corresponding to a location of a nonblank tile on C or C'. A pair of locations of a nonblank tile on C and C' is corresponding to a 2-net.

Lemma 1: Configurations C and C' of $(n^2 - 1)$-puzzle are reachable with h moves for $h \geq 2$ if and only if the 2-nets corresponding to the nonblank tiles are routable in an (n, n, h)-channel.

Proof. Suppose that configurations C and C' of $(n^2 - 1)$-puzzle are reachable with h moves for $h \geq 2$. For a sequence of moves that transforms C to C', locations in the sequence for a nonblank tile correspond to part of the wire connecting the terminals of the corresponding 2-net. Since such wires are vertex-disjoint, the 2-nets corresponding to the nonblank tiles are routable in an (n, n, h)-channel.

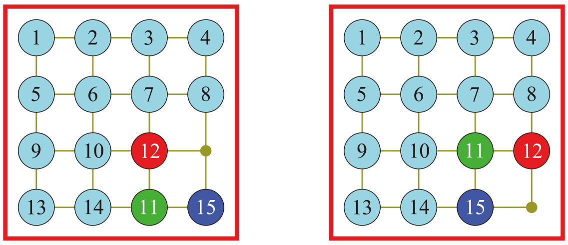
Conversely, suppose that the 2-nets corresponding to the nonblank tiles are routable in an (n, n, h)-channel with $h \geq 2$. Since the number of 2-nets is $n^2 - 1$, every wire is descending with respect to the z-coordinate, and for every layer, at most one edge of the layer is contained in the wires. Since such an edge corresponds to a move, the corresponding configurations of $(n^2 - 1)$-puzzle are reachable with h moves. □

Lemma 1 implies a polynomial time reduction from $(n^2 - 1)$-PUZZLE to 3-D CHANNEL ROUGING. Thus we



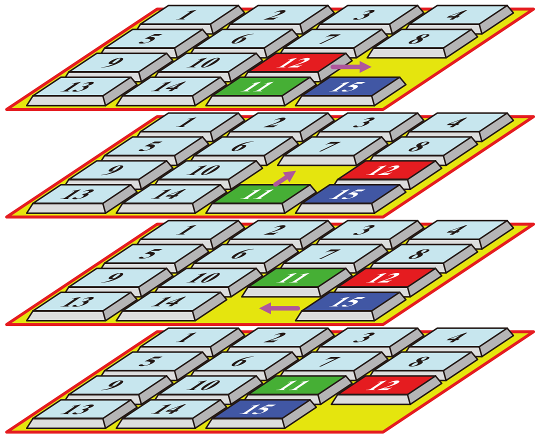
(a) C_1. (b) C_2.

Fig. 3 Initial and final configurations of 15-puzzle.

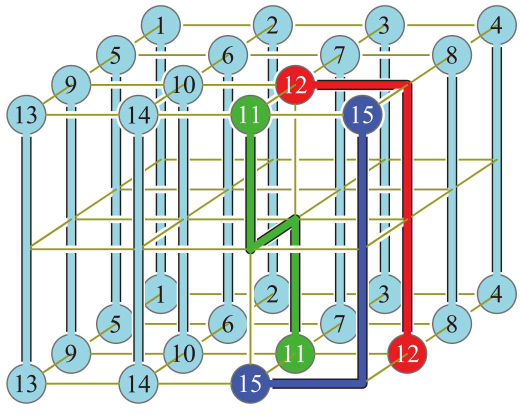


(a) Top layer. (b) Bottom layer.

Fig. 4 Corresponding 2-nets.



(a) Sequence of 3 moves that transforms C_1 to C_2.



(b) Corresponding 3-D channel routing with height 3.

Fig. 5 Correspondence between 15-puzzle and 3-D channel routing.

conclude that 3-D CHANNEL ROUTING is NP-hard by Theorem I. This completes the proof of Theorem 1.

Example 1: For initial and final configurations C_1 and C_2 of 15-puzzle shown in Fig. 3, the corresponding 2-nets are shown in Fig. 4. A sequence of 3 moves that transforms C_1 to C_2, and the corresponding 3-D channel routing with height 3 are shown in Fig. 5.

3. Sparse Instances

Let G be a $(2\sqrt{v}, 2\sqrt{v}, H)$-channel with a set

$$\mathcal{N} = \Big\{\{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\} \;\Big|\; X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle} \text{ are odd integers in } [2\sqrt{v}\,], k \in [v]\Big\}$$

of v 2-nets. $\mathcal{N}$ is said to be *sparse*. The purpose of this section is to show the following.

Theorem 2: Any sparse $\mathcal{N}$ can be routed in a $(2\sqrt{v}, 2\sqrt{v}, 3\sqrt{v}+1)$-channel using wires of length $O(\sqrt{v})$ in $O(v \log v)$ time. □

We need some preliminaries to prove the theorem.

3.1 3-D Channels

We consider a 3-D channel of height $H = 3\sqrt{v} + 1$, which is a $2\sqrt{v} \times 2\sqrt{v} \times H$ 3-D grid. Each grid point is denoted by (x, y, z) with $x, y \in [2\sqrt{v}]$ and $z \in [H]$. The column, row, and layer defined by $x = X$, $y = Y$, and $z = Z$ are called the *X-column*, *Y-row*, and *Z-layer*, respectively. The *H*-layer and 1-layer correspond to the top and bottom layers, respectively. Let $\mathcal{N} = \{N_k \mid k \in [v]\}$ be a sparse set of v 2-nets, and let $(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H)$ and $(X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)$ be the terminals of N_k ($k \in [v]$), such that $X_k^{\langle H\rangle}$, $Y_k^{\langle H\rangle}$, $X_k^{\langle 1\rangle}$, and $Y_k^{\langle 1\rangle}$ are odd, and that $(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H) \neq (X_{k'}^{\langle H\rangle}, Y_{k'}^{\langle H\rangle}, H)$ and $(X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1) \neq (X_{k'}^{\langle 1\rangle}, Y_{k'}^{\langle 1\rangle}, 1)$ if $k \neq k'$.

3.2 2-Row Channel Routings

We consider in this section the 2-row channel routing which is used as a subroutine of our 3-D channel routing algorithm. A 2-row channel of height $m+1$ is a $2m \times 2 \times (m+1)$ 3-D grid G'. Let $\mathcal{N}' = \{N'_k \mid k \in [m]\}$ be a sparse set of m 2-nets, and let $(X_k^{\langle m+1\rangle}, 1, m+1)$ and $(X_k^{\langle 1\rangle}, 1, 1)$ be the terminals of N'_k ($k \in [m]$), where $X_k^{\langle m+1\rangle}$ and $X_k^{\langle 1\rangle}$ are odd, and $X_k^{\langle m+1\rangle} \neq X_{k'}^{\langle m+1\rangle}$ and $X_k^{\langle 1\rangle} \neq X_{k'}^{\langle 1\rangle}$ if $k \neq k'$.

Lemma 2: Any sparse $\mathcal{N}'$ can be routed in G' so that no wire passes through the top layer.

Proof. Let $p_1, p_2, \ldots, p_l$ be grid points of G' such that p_i and p_{i+1} differ in just one coordinate, $i \in [l-1]$. Then, we denote by $[p_1, p_2, \ldots, p_l]$ a wire connecting p_1 and p_l obtained by connecting p_i and p_{i+1} by an axis-parallel line segment, $i \in [l-1]$. If $X_k^{\langle m+1\rangle} = X_k^{\langle 1\rangle}$ for all $k \in [m]$, the lemma clearly holds. Suppose without loss of generality that $X_1^{\langle m+1\rangle} = X_2^{\langle 1\rangle}$. Then, if $m \geq 3$, $\mathcal{N}'$ can be routed in G' using a wire defined by

$$\Big[\Big(X_1^{\langle m+1\rangle}, 1, m+1\Big), \Big(X_1^{\langle m+1\rangle}, 1, m\Big), \Big(X_1^{\langle m+1\rangle}+1, 1, m\Big), \Big(X_1^{\langle m+1\rangle}+1, 1, 1\Big), \Big(X_1^{\langle m+1\rangle}+1, 2, 1\Big), \Big(X_1^{\langle 1\rangle}, 2, 1\Big), \Big(X_1^{\langle 1\rangle}, 1, 1\Big)\Big]$$

for N_1', a wire defined by

$$\Big[\Big(X_2^{\langle m+1\rangle}, 1, m+1\Big), \Big(X_2^{\langle m+1\rangle}, 1, 2\Big), \Big(X_2^{\langle m+1\rangle}, 2, 2\Big), \Big(X_2^{\langle 1\rangle}, 2, 2\Big), \Big(X_2^{\langle 1\rangle}, 1, 2\Big), \Big(X_2^{\langle 1\rangle}, 1, 1\Big)\Big]$$

for N_2', and wires defined by

$$\Big[\Big(X_k^{\langle m+1\rangle}, 1, m+1\Big), \Big(X_k^{\langle m+1\rangle}, 1, k\Big), \Big(X_k^{\langle m+1\rangle}, 2, k\Big), \Big(X_k^{\langle 1\rangle}+1, 2, k\Big), \Big(X_k^{\langle 1\rangle}+1, 1, k\Big), \Big(X_k^{\langle 1\rangle}+1, 1, 1\Big), \Big(X_k^{\langle 1\rangle}, 1, 1\Big)\Big]$$

for N_k', $3 \leq k \leq m$. It is easy to see that the wires defined above are disjoint. If $m = 2$, $\mathcal{N}'$ can be routed in G' as shown in Fig. 6. In either case, no wire passes through the top layer.

□

The routing defined in the proof of Lemma 2 is called a *τ-routing for* $\mathcal{N}'$. It is easy to see that a τ-routing for a sparse set of v 2-nets can be computed in $O(v)$ time. An example of τ-routing is shown in Fig. 7. Now, we are ready to prove Theorem 2.

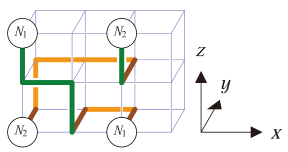


Fig. 6 A routing for a set of two 2-nets.

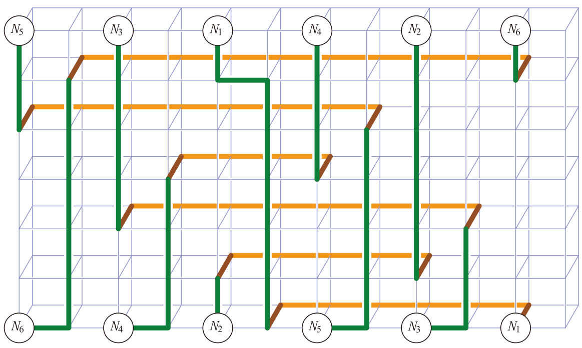


Fig. 7 A τ-routing for a set of six 2-nets.

3.3 Proof of Theorem 2

3.3.1 Virtual Terminals

We introduce in this section virtual terminals to compute a routing for a sparse set

$$\mathcal{N} = \Big\{N_k = \{(X_k^{\langle 3\sqrt{v}+1\rangle}, Y_k^{\langle 3\sqrt{v}+1\rangle}, 3\sqrt{v}+1), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\}\Big| \; X_k^{\langle 3\sqrt{v}+1\rangle}, Y_k^{\langle 3\sqrt{v}+1\rangle}, X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle} \text{ are odd integers in } [2\sqrt{v}\,], \; k \in [v] \Big\}$$

of 2-nets in a $(2\sqrt{v}, 2\sqrt{v}, 3\sqrt{v}+1)$-channel. Let $H = 3\sqrt{v}+1$, $L = 2\sqrt{v}+1$, and $M = \sqrt{v}+1$ for simplicity. By the definition of $\mathcal{N}$,

$$|\{k \in [v] \mid X_k^{\langle H\rangle} = 2j-1\}| = \sqrt{v} \text{ and} \quad (1)$$

$$|\{k \in [v] \mid X_k^{\langle 1\rangle} = 2j-1\}| = \sqrt{v}. \quad (2)$$

We use two virtual terminals $(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L)$ and $(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M)$ for each net N_k. A set of virtual terminals $\{(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L), (X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M) \mid k \in [v]\}$ is said to be *feasible* if the following conditions are satisfied:

(i) $X_k^{\langle L\rangle} = X_k^{\langle H\rangle}$ for any $k \in [v]$;
(ii) $Y_k^{\langle L\rangle} = Y_k^{\langle M\rangle}$ for any $k \in [v]$;
(iii) $X_k^{\langle M\rangle} = X_k^{\langle 1\rangle}$ for any $k \in [v]$;
(iv) $(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L) \neq (X_h^{\langle L\rangle}, Y_h^{\langle L\rangle}, L)$ if $k \neq h$;
(v) $(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M) \neq (X_h^{\langle M\rangle}, Y_h^{\langle M\rangle}, M)$ if $k \neq h$.

Lemma 3: For any sparse set $\mathcal{N}$ of 2-nets, there exists a feasible set of virtual terminals $\{(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L), (X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M) \mid k \in [v]\}$. Moreover, these virtual terminals can be computed in $O(v \log v)$ time.

Proof. For every $k \in [v]$, $Y_k^{\langle L\rangle} = Y_k^{\langle M\rangle}$ is determined as follows. Let B be a bipartite multigraph defined as follows:

$$V(B) = \{(2j-1, z) \mid j \in [\sqrt{v}\,], z \in \{1, H\}\};$$

$$E(B) = \Big\{\Big((X_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, 1)\Big) \Big| \; \{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\} \in \mathcal{N}\Big\}.$$

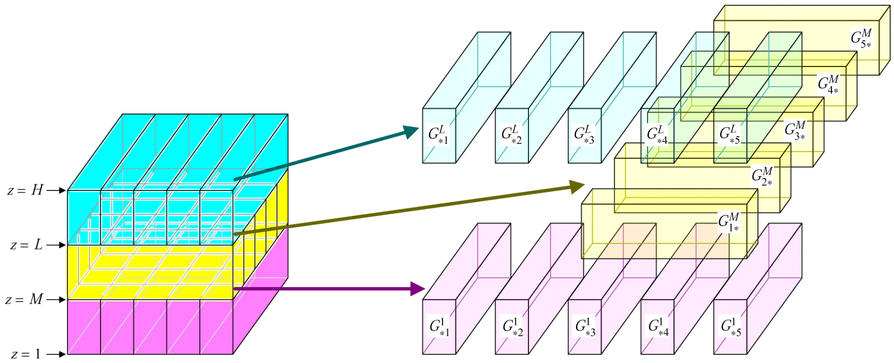
For each $j \in [\sqrt{v}\,]$, there exist exactly $\sqrt{v}$ 2-nets

$$\{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\}$$

such that $X_k^{\langle H\rangle} = 2j-1$ by (1), and exactly $\sqrt{v}$ 2-nets

$$\{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\}$$

such that $X_k^{\langle 1\rangle} = 2j-1$ by (2). Therefore, B is $\sqrt{v}$-regular. A $\sqrt{v}$-regular bipartite multigraph has a $\sqrt{v}$-edge-coloring by König's theorem [11]. Moreover, such a $\sqrt{v}$-edge-coloring can be computed in $O(|E(B)| \log |E(B)|) = O(v \log v)$ time [1], [5], [6]. Let $c : E(B) \to [\sqrt{v}\,]$ be such an edge-coloring. If c_k is the color assigned to edge $\Big((X_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, 1)\Big)$, we



(a) (10, 10, 16)-grid G. (b) subgrids G^1_{*j}, G^M_{i*}, and G^L_{*j}.

Fig. 8 An example of a (10, 10, 16)-grid G and its subgrids.

define $Y_k^{\langle L\rangle} = Y_k^{\langle M\rangle} = 2c_k - 1$. We also define $X_k^{\langle L\rangle} = X_k^{\langle H\rangle}$ and $X_k^{\langle M\rangle} = X_k^{\langle 1\rangle}$ for every $k \in [\nu]$. Then, the following set

$$\mathcal{V} = \left\{(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L), (X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M) \;\middle|\; k \in [\nu]\right\}$$

is a feasible set of virtual terminals for $\mathcal{N}$. By definition, $\mathcal{V}$ satisfies (i), (ii), and (iii). If $X_k^{\langle L\rangle} = X_h^{\langle L\rangle}$ then $X_k^{\langle H\rangle} = X_h^{\langle H\rangle}$. Thus, edges $\left((X_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, 1)\right)$ and $\left((X_h^{\langle H\rangle}, H), (X_h^{\langle 1\rangle}, 1)\right)$ of B have different colors, and we have $Y_k^{\langle L\rangle} \neq Y_h^{\langle L\rangle}$. Thus $\mathcal{V}$ satisfies (iv). If $X_k^{\langle M\rangle} = X_h^{\langle M\rangle}$ then $X_k^{\langle 1\rangle} = X_h^{\langle 1\rangle}$. Thus, edges $\left((X_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, 1)\right)$ and $\left((X_h^{\langle H\rangle}, H), (X_h^{\langle 1\rangle}, 1)\right)$ of B have different colors, and we have $Y_k^{\langle M\rangle} \neq Y_h^{\langle M\rangle}$. Thus $\mathcal{V}$ satisfies (v), and we conclude that $\mathcal{V}$ is feasible.

Since the construction of B takes $O(\nu)$ time and computation of c takes $O(\nu \log \nu)$ time, we have the lemma. □

3.3.2 Polynomial Time Algorithm

Let $G^{\langle r\rangle}_{*j}$ be a $2 \times 2\sqrt{\nu} \times (\sqrt{\nu} + 1)$-subgrid induced by a set of grid points:

$$\left\{(x, y, z) \mid x \in \{2j-1, 2j\}, y \in \left[2\sqrt{\nu}\right], r \le z \le r + \sqrt{\nu}\right\},$$

and $G^{\langle r\rangle}_{i*}$ be a subgrid induced by a set of grid points:

$$\left\{(x, y, z) \mid x \in \left[2\sqrt{\nu}\right], y \in \{2i-1, 2i\}, r \le z \le r + \sqrt{\nu}\right\}.$$

We decompose the 3-D grid into $3\sqrt{\nu}$ subgrids $G^{\langle L\rangle}_{*j}$ for $j \in [\sqrt{\nu}]$, $G^{\langle M\rangle}_{i*}$ for $i \in [\sqrt{\nu}]$, and $G^{\langle 1\rangle}_{*j}$ for $j \in [\sqrt{\nu}]$, as shown in Fig. 8. By Lemma 3, we have a feasible set of virtual terminals:

$$\mathcal{V} = \{(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L), (X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M) \mid k \in [\nu]\}.$$

We define three sets of 2-nets as follows:

$$\begin{aligned}
\mathcal{N}^{\langle L\rangle}_{*j} &= \{N_k^{\langle H,L\rangle} = \{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L)\} \mid \\
&\qquad X_k^{\langle H\rangle} = 2j - 1\},\\
\mathcal{N}^{\langle M\rangle}_{i*} &= \{N_k^{\langle L,M\rangle} = \{(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L), (X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M)\} \mid \\
&\qquad Y_k^{\langle L\rangle} = 2i - 1\}, \text{ and}\\
\mathcal{N}^{\langle 1\rangle}_{*j} &= \{N_k^{\langle M,1\rangle} = \{(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\} \mid \\
&\qquad X_k^{\langle 1\rangle} = 2j - 1\}.
\end{aligned}$$

Input $\mathcal{N} = \{N_k \mid k \in [\nu]\}$ with terminals $(X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)$ and $(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H)$ for $\forall k \in [\nu]$.
Output Routing for $\mathcal{N}$.
Step 0 **for** $\forall k \in [\nu]$,
Compute virtual terminals $(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L)$ and $(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M)$.
Step 1 **for** $\forall j \in [\sqrt{\nu}]$,
Apply τ-routing to connect $(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H)$ and $(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L)$ with $X_k^{\langle H\rangle} = X_k^{\langle L\rangle} = 2j - 1$ in G^L_{*j}.
Step 2 **for** $\forall i \in [\sqrt{\nu}]$,
Apply τ-routing to connect $(X_k^{\langle L\rangle}, Y_k^{\langle L\rangle}, L)$ and $(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M)$ with $Y_k^{\langle L\rangle} = Y_k^{\langle M\rangle} = 2i - 1$ in G^M_{i*}.
Step 3 **for** $\forall j \in [\sqrt{\nu}]$,
Apply τ-routing to connect $(X_k^{\langle M\rangle}, Y_k^{\langle M\rangle}, M)$ and $(X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)$ with $X_k^{\langle M\rangle} = X_k^{\langle 1\rangle} = 2j - 1$ in G^1_{*j}.
Step 4 **for** $\forall k \in [\nu]$,
Output a wire for N_k by concatenating three wires for N_k above.

Fig. 9 3-D channel routing algorithm.

Since $\mathcal{V}$ is feasible, the terminals of 2-nets in $\mathcal{N}^{\langle L\rangle}_{*j}$ are contained in $G^{\langle L\rangle}_{*j}$, and so $\mathcal{N}^{\langle L\rangle}_{*j}$ is routable in $G^{\langle L\rangle}_{*j}$ by using τ-routing for each $j \in [\sqrt{\nu}]$. Similarly, $\mathcal{N}^{\langle M\rangle}_{i*}$ is routable in $G^{\langle M\rangle}_{i*}$ by using τ-routing for each $i \in [\sqrt{\nu}]$, and $\mathcal{N}^{\langle 1\rangle}_{*j}$ is routable in $G^{\langle 1\rangle}_{*j}$ by using τ-routing for each $j \in [\sqrt{\nu}]$.

A wire for each 2-net N_k in $\mathcal{N}$ is obtained by concatenating three wires $N_k^{\langle H,L\rangle}$, $N_k^{\langle L,M\rangle}$, and $N_k^{\langle M,1\rangle}$.

Our 3-D channel routing algorithm is shown in Fig. 9. It is straightforward that $\mathcal{N}$ is routed in a 3-D channel of height $3\sqrt{\nu}+1$. Since the length of every wire of a τ-routing is at most $3\sqrt{\nu} + 4$, the maximum wire length of our 3-D channel routing algorithm is at most $9\sqrt{\nu} + 12$.

It should be noted that the time complexity of our 3-D channel routing algorithm is $O(\nu \log \nu)$, since Step 0 takes

$O(v \log v)$ time, and other steps take $O(v)$ time as easily seen. This completes the proof of Theorem 2.

4. Lower Bounds

We investigate in this section some lower bounds for the height of 3-D channel routing. We assume for simplicity that G is an (S, S, H)-channel, and

$$\mathcal{N} = \{N_k = \{(x_k^t, y_k^t, H), (x_k^b, y_k^b, 1)\} \mid k \in [v]\}$$

is a set of v 2-nets, where $v < S^2$, and $H \geq 2$.

4.1 Densities

Our first lower bound is the *layer density* $\Delta_{\rm lay}(\mathcal{N})$ which is defined as follows:

$$\Delta_{\rm lay}(\mathcal{N}) = \frac{\sum_{k=1}^{v}\left(|x_k^t - x_k^b| + |y_k^t - y_k^b|\right)}{S^2 - v}.$$

Theorem 3: If $\mathcal{N}$ is routable in G then $H \geq \lceil \Delta_{\rm lay}(\mathcal{N}) \rceil$.

Proof. Since the length of a shortest path connecting terminals of N_k is $|x_k^t - x_k^b| + |y_k^t - y_k^b| + H - 1$, any routing of N_k in G contains $|x_k^t - x_k^b| + |y_k^t - y_k^b| + H$ grid points. Thus,

$$\sum_{k=1}^{v}\left(|x_k^t - x_k^b| + |y_k^t - y_k^b| + H\right) \leq S^2 H,$$

and we have $H \geq \Delta_{\rm lay}(\mathcal{N})$. Since H is an integer, $H \geq \lceil \Delta_{\rm lay}(\mathcal{N}) \rceil$ and we have the theorem. □

In Figs. 10 and 11, terminals of a net N_k are denoted by k. It is easy to see that $\Delta_{\rm lay}(\mathcal{N}_a) = 28$, and $\Delta_{\rm lay}(\mathcal{N}_b) = 3/5$.

Our second lower bound is the *global density* $\Delta_{\rm glo}(\mathcal{N})$ which is defined as follows. Let $R_1, R_2, \ldots, R_S$ be the rows of G, and $C_1, C_2, \ldots, C_S$ be the columns of G (See Figs. 10 and 11). For any $i, j \in [v]$, let

$$\begin{aligned}
T^t(R_i) &= \{(x_k^t, y_k^t, H) \mid k \in [v], y_k^t = i\},\\
T^b(R_i) &= \{(x_k^b, y_k^b, 1) \mid k \in [v], y_k^b = i\},\\
\mathcal{N}(R_i) &= \{N_k \mid k \in [v], (y_k^t - i)(y_k^b - i) < 0\},\\
T^t(C_j) &= \{(x_k^t, y_k^t, H) \mid k \in [v], x_k^t = j\},\\
T^b(C_j) &= \{(x_k^b, y_k^b, 1) \mid k \in [v], x_k^b = j\}, \text{ and}\\
\mathcal{N}(C_j) &= \{N_k \mid k \in [v], (x_k^t - j)(x_k^b - j) < 0\}.
\end{aligned}$$

The following is immediate.

Lemma 4: A wire of any net in $\mathcal{N}(R_i)$ $[\mathcal{N}(C_j)]$ contains a vertex of R_i $[C_j]$. □

Let $d(R_i)$ $[d(C_j)]$ be the sum of the number of terminals on R_i $[C_j]$ and the number of 2-nets which have a terminal on both sides of R_i $[C_j]$, that is,

$$d(R_i) = |T^t(R_i)| + |T^b(R_i)| + |\mathcal{N}(R_i)|, \text{ and} \quad (3)$$

$$d(C_j) = |T^t(C_j)| + |T^b(C_j)| + |\mathcal{N}(C_j)|. \quad (4)$$

Notice that

$$T^t(R_i) \cup T^b(R_i) \subseteq V(R_i), \text{ and} \quad (5)$$

$$T^t(C_j) \cup T^b(C_j) \subseteq V(C_j). \quad (6)$$

We define that:

$$\Delta_{\rm glo}(\mathcal{N}) = \max\left\{\frac{\max\{d(R_i) \mid i \in [S]\}}{S}, \frac{\max\{d(C_j) \mid j \in [S]\}}{S}\right\}.$$

Theorem 4: If $\mathcal{N}$ is routable in G then $H \geq \lceil \Delta_{\rm glo}(\mathcal{N}) \rceil$.

Proof. From Lemma 4, (3), and (5), we have $d(R_i) \leq |V(R_i)| = SH$ for any $i \in [v]$, since wires are vertex-disjoint. Similarly, we have $d(C_j) \leq SH$ for any $j \in [v]$. Thus, we have

$$H \geq \frac{d(R_i)}{S}, \frac{d(C_j)}{S}$$

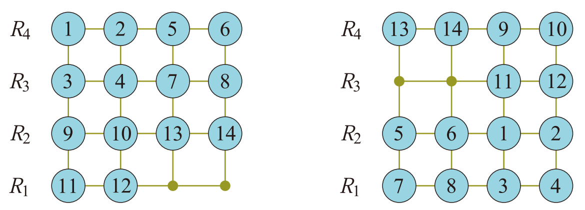
for any $i, j \in [S]$, and we have the theorem. □

In Figs. 10 and 11, let t_k and b_k be the terminals of N_k on the top and bottom layers, respectively. In Fig. 10, $\{t_9, t_{10}, t_{13}, t_{14}, b_5, b_6, b_1, b_2\} \subseteq V(R_2)$, and for each $k \in \{3, 4, 7, 8, 11, 12\}$, t_k and b_k are on different sides of R_2. Therefore, we have $d(R_2) = 14$. Since

$$\begin{aligned}
&\max\{\max\{d(C_j) \mid j \in [S]\}, \max\{d(R_i) \mid i \in [S]\}\}\\
&= d(R_2)\\
&= 14,
\end{aligned}$$

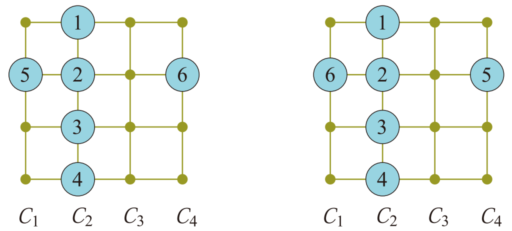
we have $\Delta_{\rm glo}(\mathcal{N}_a) = 14/4$. In Fig. 11, terminals t_k and b_k for each $k \in \{1, 2, 3, 4\}$ are on C_2, and terminals t_k and b_k for each $k \in \{5, 6\}$ are on different sides of C_2. Therefore,

$$d(C_2) = |\{t_k, b_k \mid k \in \{1, 2, 3, 4\}\} \cup \{N_5, N_6\}| = 10,$$



(a) Top layer. (b) Bottom layer.

Fig. 10 $\mathcal{N}_a$ such that $\Delta_{\rm lay}(\mathcal{N}_a)$ dominates $\Delta_{\rm glo}(\mathcal{N}_a)$ and $\Delta_{\rm loc}(\mathcal{N}_a)$.



(a) Top layer. (b) Bottom layer.

Fig. 11 $\mathcal{N}_b$ such that $\Delta_{\rm glo}(\mathcal{N}_b)$ dominates $\Delta_{\rm lay}(\mathcal{N}_b)$ and $\Delta_{\rm loc}(\mathcal{N}_b)$.

and we have $\Delta_{\mathrm{glo}}(\mathcal{N}_b) \geq 10/4$.

Our final lower bound is the *local density* $\Delta_{\mathrm{loc}}(\mathcal{N})$ which is defined as follows. Let Q be a cycle on top layer L_t, Q' be the corresponding cycle on bottom layer L_b, and Q_i be the corresponding cycle on the i-th layer defined by $z = i$. Notice that $V(Q_i) = \{(x, y, i) \mid (x, y, H) \in V(Q)\}$, $Q_H = Q$, and $Q_1 = Q'$. Let $T(Q)$ be the set of terminals on Q, $T(Q')$ be the set of terminals on Q', $\mathcal{N}_Q$ be the set of nets which have a terminal inside of Q on L_t and a terminal outside of Q' on L_b, and $\mathcal{N}(Q')$ be the set of nets which have a terminal outside of Q on L_t and a terminal inside of Q' on L_b. The following is immediate.

Lemma 5: A wire of any net in N_Q $[Q']$ contains a vertex of $\bigcup_{i=1}^{H} V(Q_i)$. □

Let $d(Q)$ $[d(Q')]$ be the sum of the number of terminals on Q $[Q']$ and the number of 2-nets which have a terminal inside of Q $[Q']$ on L_t $[L_b]$, and a terminal outside of Q' $[Q]$ on L_b $[L_t]$, that is,

$$d(Q) = |\mathcal{N}(Q)| + |T(Q)|, \text{ and} \tag{7}$$
$$d(Q') = |\mathcal{N}(Q')| + |T(Q')|. \tag{8}$$

Notice that

$$T(Q) \subseteq V(Q), \text{ and} \tag{9}$$
$$T(Q') \subseteq V(Q'). \tag{10}$$

We define that:

$$\Delta_{\mathrm{loc}}(\mathcal{N}) = \max\left\{ \frac{d(Q) + d(Q')}{|V(Q)|} \,\middle|\, Q : \text{ a cycle on } L_t \right\}.$$

Theorem 5: If $\mathcal{N}$ is routable in G then $H \geq \lceil \Delta_{\mathrm{loc}}(\mathcal{N}) \rceil$.

Proof. From Lemma 5, (7), (8), (9), and (10), we have

$$d(Q) + d(Q') \leq \left| \bigcup_{i=1}^{H} V(Q_i) \right| = H|V(Q)|,$$

since wires are vertex-disjoint. Thus, we have

$$H \geq \frac{d(Q) + d(Q')}{|V(Q)|}$$

for any cycle Q on the top layer, and we have the theorem. □

In Fig. 10, if $I(Q)$ is the set of inner vertices of Q on L_t, we have $|I(Q)| < |V(Q)|/2$, since $S = 4$. Therefore, $d(Q) \leq |V(Q)| + |I(Q)| < 3|V(Q)|/2$. Similarly, we have $d(Q') < 3|V(Q')|/2$. Thus, we have $\Delta_{\mathrm{loc}}(\mathcal{N}_a) < 3/2 + 3/2 = 3$. In Fig. 11, we have $d(Q) < |V(Q)|$ and $d(Q') < |V(Q')|$ for any cycle Q and Q' on L_t and L_b, respectively. Therefore, $\Delta_{\mathrm{loc}}(\mathcal{N}_b) < 2$.

4.2 Comparisons

We can show that there are instances $\mathcal{N}_{\mathrm{lay}}$, $\mathcal{N}_{\mathrm{glo}}$, and $\mathcal{N}_{\mathrm{loc}}$ such that $\Delta_{\mathrm{lay}}(\mathcal{N}_{\mathrm{lay}})$ dominates $\Delta_{\mathrm{glo}}(\mathcal{N}_{\mathrm{lay}})$ and $\Delta_{\mathrm{loc}}(\mathcal{N}_{\mathrm{lay}})$, $\Delta_{\mathrm{glo}}(\mathcal{N}_{\mathrm{glo}})$ dominates $\Delta_{\mathrm{lay}}(\mathcal{N}_{\mathrm{glo}})$ and $\Delta_{\mathrm{loc}}(\mathcal{N}_{\mathrm{glo}})$, and $\Delta_{\mathrm{loc}}(\mathcal{N}_{\mathrm{loc}})$ dominates $\Delta_{\mathrm{lay}}(\mathcal{N}_{\mathrm{loc}})$ and $\Delta_{\mathrm{loc}}(\mathcal{N}_{\mathrm{loc}})$.

For $\mathcal{N}_a$ in Fig. 10, $\Delta_{\mathrm{lay}}(\mathcal{N}_a)$ dominates $\Delta_{\mathrm{glo}}(\mathcal{N}_a)$ and $\Delta_{\mathrm{loc}}(\mathcal{N}_a)$, since $\Delta_{\mathrm{lay}}(\mathcal{N}_a) = 28$, $\Delta_{\mathrm{glo}}(\mathcal{N}_a) = 14/4$, and $\Delta_{\mathrm{loc}}(\mathcal{N}_a) < 3$ as we have calculated. For $\mathcal{N}_b$ in Fig. 11, $\Delta_{\mathrm{glo}}(\mathcal{N}_b)$ dominates $\Delta_{\mathrm{lay}}(\mathcal{N}_b)$ and $\Delta_{\mathrm{loc}}(\mathcal{N}_b)$, since $\Delta_{\mathrm{glo}}(\mathcal{N}_b) \geq 10/4$, $\Delta_{\mathrm{lay}}(\mathcal{N}_b) \leq 1$, and $\Delta_{\mathrm{loc}}(\mathcal{N}_b) < 2$ as we have calculated.

The proof of Theorem 8 shown in the next section provides a set of nets $\mathcal{N}$ such that $\Delta_{\mathrm{loc}}(\mathcal{N})$ dominates $\Delta_{\mathrm{glo}}(\mathcal{N})$ and $\Delta_{\mathrm{lay}}(\mathcal{N})$ if ν is sufficiently large.

It is interesting to note that $\Delta_{\mathrm{loc}}(\mathcal{N})$ asymptotically dominates $\Delta_{\mathrm{glo}}(\mathcal{N})$ for any $\mathcal{N}$ as shown in the following.

Theorem 6: $\Delta_{\mathrm{glo}}(\mathcal{N}) = O(\Delta_{\mathrm{loc}}(\mathcal{N}))$ for any instance if the layer is square.

Proof. For any $x, y \in [S]$, let $X_{x,h}$ and $Y_{y,h}$ be cycles induced by vertex sets

$$\begin{aligned} &V(X_{x,h}) \\ &\quad = \{(j, 1, h) \mid 1 \leq j \leq x\} \cup \{(j, S, h) \mid 1 \leq j \leq x\} \cup \\ &\qquad \{(1, i, h) \mid 1 \leq i \leq S\} \cup \{(x, i, h) \mid 1 \leq i \leq S\}, \text{ and} \\ &V(Y_{y,h}) \\ &\quad = \{(1, i, h) \mid 1 \leq i \leq y\} \cup \{(S, i, h) \mid 1 \leq i \leq y\} \cup \\ &\qquad \{(j, 1, h) \mid 1 \leq j \leq S\} \cup \{(j, y, h) \mid 1 \leq j \leq S\}, \end{aligned}$$

respectively. By definition, we have

$$\begin{aligned} &d(X_{x,h}) \geq d(C_x), \;\; d(Y_{y,h}) \geq d(R_y), \text{ and} \\ &|V(X_{x,h})|, |V(Y_{y,h})| \leq 4S. \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\Delta_{\mathrm{glo}}(\mathcal{N}) \\ &\quad = \max\left\{ \frac{\max\{d(R_i) \mid i \in [S]\}}{S}, \frac{\max\{d(C_j) \mid j \in [S]\}}{S} \right\} \\ &\quad \leq \max\left\{ \max\left\{ \frac{d(X_{x,H}) + d(X_{x,1})}{|V(X_{x,H})|/4} \,\middle|\, x \in [S] \right\}, \right. \\ &\qquad\qquad \left. \max\left\{ \frac{d(Y_{y,H}) + d(Y_{y,1})}{|V(Y_{y,H})|/4} \,\middle|\, y \in [S] \right\} \right\} \\ &\quad \leq \max\left\{ \frac{d(Q) + d(Q')}{|V(Q)|/4} \,\middle|\, Q : \text{a cycle on } L_t. \right\} \\ &\quad = 4\Delta_{\mathrm{loc}}(\mathcal{N}), \end{aligned}$$

and we obtain the theorem. □

4.3 Sparse Instances

Suppose that G is a $(2\sqrt{\nu}, 2\sqrt{\nu}, H)$-channel with a sparse set $\mathcal{N} = \{N_k \mid i \in [\nu]\}$ of 2-nets, and $N_k = \{(x_k^t, y_k^t, H), (x_k^b, y_k^b, 1)\}$, where x_k^t, y_k^t, x_k^b, and y_k^b are odd integers. We have shown in Sect. 3 that any sparse instance $\mathcal{N}$ is routable in G if $H \geq 3\sqrt{\nu} + 1$.

It follows from Theorem 6 above and Theorem 7 below that $\Delta_{\mathrm{loc}}(\mathcal{N})$ asymptotically dominates $\Delta_{\mathrm{lay}}(\mathcal{N})$ and $\Delta_{\mathrm{glo}}(\mathcal{N})$ for sparse instances.

Theorem 7: $\Delta_{\text{lay}}(\mathcal{N}) = O(\Delta_{\text{glo}}(\mathcal{N}))$ for any sparse instance.

Proof. It is easy to see the following.

$$\begin{aligned} 3\nu\Delta_{\text{lay}}(\mathcal{N}) &= \sum_{k=1}^{\nu}\left(|x_k^t - x_k^b| + |y_k^t - y_k^b|\right) \\ &\leq \sum_{j=1}^{2\sqrt{\nu}} d(C_j) + \sum_{i=1}^{2\sqrt{\nu}} d(R_i) \\ &\leq 2\sum_{i=1}^{2\sqrt{\nu}} 2\sqrt{\nu}\Delta_{\text{glo}}(\mathcal{N}) \\ &= 8\nu\Delta_{\text{glo}}(\mathcal{N}). \end{aligned}$$

It follows that $\Delta_{\text{lay}}(\mathcal{N}) \leq \frac{8}{3}\Delta_{\text{glo}}(\mathcal{N})$, and we have the theorem. □

On the other hand, there are sparse instances $\mathcal{N}$ such that neither $\Delta_{\text{lay}}(\mathcal{N})$ nor $\Delta_{\text{glo}}(\mathcal{N})$ asymptotically dominates $\Delta_{\text{loc}}(\mathcal{N})$ as shown below.

Theorem 8: There exist sparse instances such that $\Delta_{\text{loc}}(\mathcal{N}) = \omega(\Delta_{\text{glo}}(\mathcal{N}))$.

Proof. Let Q_1 and Q_2 be disjoint square cycles on L_t such that neither is inside of the other, and $|V(Q_1)| = |V(Q_2)| = 8\lfloor\sqrt[4]{\nu}\rfloor - 4$. (See Fig. 12.) Suppose that each 2-net with a terminal inside Q_1 $[Q_2]$ on L_t has the other inside Q_2' $[Q_1']$ on L_b, and for every other 2-net, the terminals on L_t and L_b have the same x- and y-coordinates. Since $d(Q_1) = d(Q_2) = \lfloor\sqrt[4]{\nu}\rfloor^2 + 2\lfloor\sqrt[4]{\nu}\rfloor - 1$, $\Delta_{\text{loc}}(\mathcal{N}) = \Omega(\lfloor\sqrt[4]{\nu}\rfloor)$. On the other hand, $\Delta_{\text{glo}}(\mathcal{N}) \leq 2$ as easily seen, and we have the theorem. □

Finally, we show the following which complements Theorem 2.

Theorem 9: There exists a sparse set of 2-nets $\mathcal{N}$ that cannot be routed in a $(2\sqrt{\nu}, 2\sqrt{\nu}, 2\sqrt{\nu}/3 - 1)$-channel.

Proof. For $i \in [\sqrt{\nu}]$, $j \in [\sqrt{\nu}]$, and $k = (j-1)\sqrt{\nu} + i$, define that

$$X_k^{\langle 1\rangle} = 2j - 1,$$

$$X_k^{\langle H\rangle} = \begin{cases} 2j + \sqrt{\nu} - 1 & \text{if} \quad j \leq \sqrt{\nu} \\ 2j - \sqrt{\nu} - 1 & \text{if} \quad j \geq \sqrt{\nu} + 1, \end{cases}$$

$$Y_k^{\langle 1\rangle} = 2i - 1,$$

$$Y_k^{\langle H\rangle} = \begin{cases} 2i + \sqrt{\nu} - 1 & \text{if} \quad i \leq \sqrt{\nu}, \text{ and} \\ 2i - \sqrt{\nu} - 1 & \text{if} \quad i \geq \sqrt{\nu} + 1. \end{cases}$$

By the definitions of $X_k^{\langle 1\rangle}$, $X_k^{\langle H\rangle}$, $Y_k^{\langle 1\rangle}$, and $Y_k^{\langle H\rangle}$, we have

$$\begin{aligned} \left|X_k^{\langle 1\rangle} - X_k^{\langle H\rangle}\right| &= \sqrt{\nu}, \text{ and} \\ \left|Y_k^{\langle 1\rangle} - Y_k^{\langle H\rangle}\right| &= \sqrt{\nu}, \end{aligned}$$

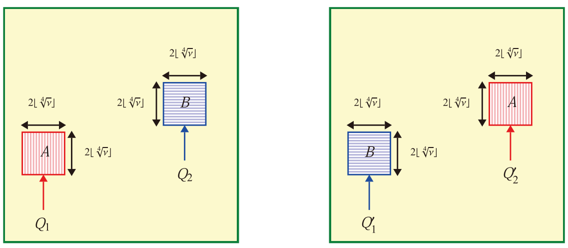
i.e.,

$$\sum_{k=1}^{\nu}\left(\left|X_k^{\langle 1\rangle} - X_k^{\langle H\rangle}\right| + \left|Y_k^{\langle 1\rangle} - Y_k^{\langle H\rangle}\right|\right) = 2\nu\sqrt{\nu}. \tag{11}$$

Let $\mathcal{N} = \{N_k \mid k \in [\nu]\}$ be a set of ν 2-nets such that $N_k = \{(X_k^{\langle H\rangle}, Y_k^{\langle H\rangle}, H), (X_k^{\langle 1\rangle}, Y_k^{\langle 1\rangle}, 1)\}$. From (11), we have

$$\Delta_{\text{lay}}(\mathcal{N}) = \frac{2\nu\sqrt{\nu}}{4\nu - \nu} = \frac{2\sqrt{\nu}}{3}.$$

Thus, $\mathcal{N}$ cannot be routed in a $(2\sqrt{\nu}, 2\sqrt{\nu}, 2\sqrt{\nu}/3 - 1)$-channel, and we have the theorem. □



(a) Top layer. (b) Bottom layer.

Fig. 12 An example of a set $\mathcal{N}$ such that $\Delta_{\text{loc}}(\mathcal{N})$ dominates $\Delta_{\text{glo}}(\mathcal{N})$ and $\Delta_{\text{lay}}(\mathcal{N})$.

5. Concluding Remarks

We have shown that 3-D CHANNEL ROUTING is NP-hard. In fact, we can show that 3-D CHANNEL ROUTING is NP-complete. It is shown in [20], [21] that 3-D channel routing is indeed in NP.

The Manhattan model is one of the most popular 2-D channel routing models for practitioners. Szymanski [18] proved that the corresponding decision problem is NP-hard, while the complexity of the problem for 2-nets has been open as mentioned in [13]. The knock-knee model is another popular 2-D channel routing model. Sarrafzardeah [17] proved that the corresponding decision problem is NP-hard, while the complexity of the problem for 2-nets is also open. It is interesting to note that 3-D CHANNEL ROUTING is NP-hard even for 2-nets as we have shown in this paper.

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