Weighted Dominating Sets and Induced Matchings in Orthogonal Ray Graphs

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***Abstract*—An orthogonal ray graph is a graph such that for each vertex, there exists an axis-parallel rays (closed half-lines) in the plane, and two vertices are adjacent if and only if the corresponding rays intersect. A 2-directional orthogonal ray graph is an orthogonal ray graph such that the corresponding ray of each vertex is a rightward ray or a downward ray. We recently showed in [12] that the weighted dominating set problem can be solved in $O(n^4 \log n)$ time for vertex-weighted 2-directional orthogonal ray graphs by using a new parameter, boolean-width of graphs, where n is the number of vertices in a graph. We improve the result by showing an $O(n^3)$-time algorithm to solve the problem, based on a direct dynamic programming approach. We also show that the weighted induced matching problem can be solved in $O(m^6)$ time for edge-weighted orthogonal ray graphs, where m is the number of edges in a graph, closing the gap posed in [12].**

I. Introduction

A bipartite graph G with a bipartition (U, V) is called an *orthogonal ray graph* [10] (*ORG* for short) if there exist a set of disjoint horizontal rays (closed half-lines) R_u, $u \in U$, in the xy-plane, and a set of disjoint vertical rays R_v, $v \in V$, such that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if R_u and R_v intersect. A set $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$ is called an *orthogonal ray representation* of G. ORGs have been introduced in connection with the defect-tolerant design of nano-circuits [9]. An ORG is called a 3*-directional orthogonal ray graph* (3*-DORG* for short) if every vertical ray R_v, $v \in V$, has the same direction. A 3-DORG is called a 2*-directional orthogonal ray graph* (2*-DORG* for short) if every horizontal ray R_u, $u \in U$, has the same direction.

The class of 2-DORGs has been well studied, and various characterizations with an $O(n^2)$-time recognition algorithm have been known [9], [10], where n is the number of vertices in a graph. Also, some problems are known to be solvable in polynomial time for 2-DORGs (See [4], [7], [9], [11] for example, and [12] for survey).

We recently showed in [12] that the weighted dominating set problem can be solved in $O(n^4 \log n)$ time for 2-DORGs by using a new parameter, boolean-width of graphs, which is introduced in [1], [2]. Based on the result, we show in Section 3 a dynamic programming algorithm that solves the weighted dominating set problem in $O(n^3)$ time for vertex-weighted 2-DORGs.

We also showed in [12] that the weighted induced matching problem can be solved in $O(m^4)$ time for 3-DORGs, where m is the number of edges in a graph. We show in Section 4 that the weighted induced matching problem can be solved in $O(m^6)$ time for ORGs, closing the gap posed in [12].

We note that the complexity of the weighted dominating set problem for ORGs and 3-DORGs still remains open, whereas the problems can be solved in polynomial time if we have orthogonal ray representations of graphs. See [12] for details.

II. Preliminaries

All graphs considered in this paper are finite, simple, and undirected. For a graph G, let $V(G)$ and $E(G)$ denote the set of vertices and edges, respectively. Let $n = |V(G)|$ and $m = |E(G)|$. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) \mid (u, v) \in E(G)\}$, and the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N_G(v)$. The closed neighborhood of a vertex set $S \subseteq V(G)$ is $N_G[S] = \bigcup_{v \in S} N_G[v]$. If no confusion arises, we will omit the index.

III. Dominating Set Problem

A vertex v in a graph G is said to *dominate* all vertices in $N[v]$. A vertex set $D \subseteq V(G)$ is said to *dominate* $v \in V(G)$ if D has at least one vertex dominating v. A vertex set $D \subseteq V(G)$ is called a *dominating set* of G if every vertex in G is dominated by D. The (*weighted*) *dominating set problem* is to find a dominating set with minimum cardinality (weight) in a given (vertex-weighted) graph. The dominating set problem is one of the most basic and well-studied problems in graph algorithms, and has many applications in various areas including computer networks. Previous works of the problem for graphs related to ORGs can be found in [12].

In this section, we show a dynamic programming algorithm that solves the weighted dominating set problem in $O(n^3)$ time for 2-DORGs.

Let $c : V(G) \to \mathbb{R}$ be a weight (or cost) function of a graph G, where $\mathbb{R}$ is a set of real numbers, and let $c(v)$ denotes the *weight* of a vertex v in G. For a vertex set $D \subseteq V(G)$, let $c(D) = \sum_{v \in D} c(v)$ be the weight of D. It is shown in [5] that any algorithm to find a minimum-weight dominating set for graphs with non-negative weights can be extended without loss of efficiency to the algorithm for graphs with negative weights. Thus in the rest of this paper, we assume that $c(v)$ is non-negative for every vertex v in G.

For convenience, we use $S + v$ and $S - v$ instead of $S \cup \{v\}$ and $S \setminus \{v\}$, respectively, where S is a vertex set of a graph G and

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v is a vertex of G. For a family of vertex sets $\{S_1, S_2, \ldots, S_k\}$ of G, we also use $\min\{S_1, S_2, \ldots, S_k\}$ to denote a set S_i with minimum weight.

Let G be a 2-DORG with a bipartition (U, V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$. We assume without loss of generality that every R_u, $u \in U$ is a rightward ray, and every R_v, $v \in V$ is a downward ray. It is known that such an orthogonal ray representation of a 2-DORG can be obtained in $O(n^2)$ time [10]. Let (x_w, y_w) be the endpoint of R_w, $w \in V(G)$. We can assume that every x_w is distinct and every y_w is distinct [11]. We refer to x_w and y_w as the x- and y-coordinate of w, respectively. Notice that for any $u \in U$ and $v \in V$, $(u, v) \in E(G)$ if and only if $x_u < x_v$ and $y_u < y_v$.

Let G be a 2-DORG with a bipartition (U, V) and a non-negative weight function c, and let $(w_1, w_2, \ldots, w_n)$ be the total ordering of $V(G)$ such that for any w_i and w_j, $i < j$ if and only if $x_{w_i} < x_{w_j}$. For convenience of algorithm description, we add two isolated dummy vertices w_0 and w_{n+1} with weight 0. Let w_0 be a vertex in U, and we denote it by u_d. Let w_{n+1} be a vertex in V, and we denote it by v_d. We define for any $i \in \{0, 1, \ldots, n\}$ that

$$W_i = \{w_j \in V(G) \cup \{u_d, v_d\} \mid j \le i\},$$
$$U_i = W_i \cap U, \text{ and}$$
$$V_i = \overline{W_i} \cap V,$$

where $\overline{W_i} = (V(G) \cup \{u_d, v_d\}) \setminus W_i$. Notice that $u_d \in U_i$ and $v_d \in V_i$ for any i.

For a vertex set $S \subseteq W_i$, let $u_S \in S \cap U_i$ be the vertex with minimum y-coordinate among $S \cap U_i$. Notice that any vertex in $\overline{W_i}$ dominated by S must be dominated by u_S, that is, $N[S] \cap \overline{W_i} = N[u_S] \cap \overline{W_i}$. Based on this observation, we show in [12] that the weighted dominating set problem can be solved in $O(n^4 \log n)$ time for 2-DORGs. This observation is also a basis of the algorithm in this paper. We refer to u_S as the *representative* of S.

A pair (S, v) of a vertex set $S \subseteq W_i$ and a vertex $v \in V_i$ is said to *dominate* W_i if all vertices in W_i are dominated by S or v, that is, $W_i \subseteq N[S + v]$.

In the algorithm, we use the two-dimensional table D_i for each $i \in \{0, 1, \ldots, n\}$ that has index set $U_i \times V_i$. The contents of $D_i[u][v]$ for every $u \in U_i$ and $v \in V_i$ are defined as follows:

$$\mathcal{S}_i[u][v] = \left\{ S \subseteq W_i \;\middle|\; \begin{array}{l} u \text{ is the representative of } S \\ \text{and } (S, v) \text{ dominates } W_i. \end{array} \right\}.$$
$$D_i[u][v] = \min\{\mathcal{S}_i[u][v]\}.$$

Notice that $u_d \in D_i[u][v]$ for any $i \in \{0, 1, \ldots, n\}$, $u \in U_i$, and $v \in V_i$, since u_d is isolated. Notice also that $D_n[u][v_d] - u_d$ with minimum weight over all $u \in U_n$ is the minimum-weight dominating set of the input 2-DORG. We compute $D_{i+1}[u][v]$ for every $u \in U_{i+1}$ and $v \in V_{i+1}$ by the following relationship among these data structures, which are proved in the following sections. In the rest of this paper, we use ∞ to denote a set of sufficiently large weight so that $D_i[u][v] = \infty$ means that there exists no such vertex set in the input graph.

Algorithm 1: Finding a minimum-weight dominating set in a 2-DORG.

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Input: An orthogonal ray representation of a 2-DORG G.
Output: A minimum-weight dominating set in G.
Add two isolated dummy vertices u_d and v_d;
D_0[u_d][v] ← u_d for all v ∈ V;
D_i[u][v] ← ∞ for all i ∈ {1, 2, ..., n}, u ∈ U_i, and v ∈ V_i;
for i ← 1 to n do
    foreach u ∈ U_i and v ∈ V_{i+1} do
        if w_{i+1} ∈ U then
            u' ← w_{i+1};
            if y_{u'} < y_u then
                D_{i+1}[u'][v] ← min{D_{i+1}[u'][v], D_i[u][v]+u'};
                if (u', v) ∈ E(G) then
                    D_{i+1}[u][v] ← D_i[u][v];
                else
                    D_{i+1}[u][v] ← ∞;
            else
                if (u', v) ∈ E(G) then
                    D_{i+1}[u][v] ← D_i[u][v];
                else
                    D_{i+1}[u][v] ← D_i[u][v] + u';
        if w_{i+1} ∈ V then
            v' ← w_{i+1};
            if y_{v'} < y_v then
                if (u, v') ∈ E(G) then
                    D_{i+1}[u][v] ← D_i[u][v];
                else
                    D_{i+1}[u][v] ← D_i[u][v] + v';
            else
                if (u, v') ∈ E(G) then
                    D_{i+1}[u][v] ← min{D_i[u][v], D_i[u][v'] + v'};
                else
                    D_{i+1}[u][v] ← D_i[u][v'] + v';
return D_n[u][v_d] − u_d with minimum weight over all u ∈ U_n;
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Lemma 1. *Suppose $w_{i+1} \in U$. Then, $D_{i+1}[u][v]$ is*

- $\min_{u'' \in U_i}\{D_i[u''][v] + u' \mid y_{u'} < y_{u''}\}$, *if $u = u'$,*
- $D_i[u][v]$, *if $u \neq u'$, $y_{u'} < y_u$, and $(u', v) \in E(G)$,*
- ∞, *if $u \neq u'$, $y_{u'} < y_u$, and $(u', v) \notin E(G)$,*
- $D_i[u][v]$, *if $u \neq u'$, $y_{u'} > y_u$, and $(u', v) \in E(G)$,*
- $D_i[u][v] + u'$, *if $u \neq u'$, $y_{u'} > y_u$, and $(u', v) \notin E(G)$,*

where $u' = w_{i+1}$. ■

Lemma 2. *Suppose $w_{i+1} \in V$. Then, $D_{i+1}[u][v]$ is*

- $D_i[u][v]$, *if $y_{v'} < y_v$ and $(u, v') \in E(G)$,*
- $D_i[u][v] + v'$, *if $y_{v'} < y_v$ and $(u, v') \notin E(G)$,*
- $\min\{D_i[u][v], D_i[u][v'] + v'\}$, *if $y_{v'} > y_v$ and $(u, v') \in E(G)$,*
- $D_i[u][v'] + v'$, *if $y_{v'} > y_v$ and $(u, v') \notin E(G)$,*

where $v' = w_{i+1}$. ■

Lemmas 1 and 2 establish Algorithm 1 shown above by using dynamic programming techniques for computing $D_i[u][v]$ for each $u \in U_i$ and $v \in V_i$ in the increasing order of $i \in \{1, 2, \ldots, n\}$.

Theorem 3. *Algorithm 1 solves the weighted dominating set problem in $O(n^3)$ time for 2-DORGs.*

Proof: Correctness of the algorithm is shown by Lemmas 1 and 2. Since the algorithm consists of three nested loops and each loop index ($i \in \{1, 2, \ldots, n\}$, $u \in U_i$, and $v \in V_{i+1}$) takes at most $n+1$ values, the algorithm runs in $O(n^3)$ time. ■

A. Proof of Lemma 1

We will compute $D_{i+1}[u][v]$ from D_i. Recall that $w_{i+1} \in U$, and then, let $u' = w_{i+1}$. We distinguish two cases.

Case 1. $u = u'$: Notice that $u' \in D_{i+1}[u'][v]$ by definition, and $D_{i+1}[u'][v]$ has no vertex in U_i whose y-coordinate is lower than that of u', for otherwise u' is no longer the representative of the vertex set. We have the followings.

Claim 4. *If $w_{i+1} = u' \in U$, then*

$$D_{i+1}[u'][v] - u' = D_i[u''][v],$$

where u'' is the representative of $D_{i+1}[u'][v] - u'$, that is, the vertex in $D_{i+1}[u'][v] \cap U$ with the second minimum y-coordinate.

Proof: Since W_i has no vertex dominated by u', $(D_{i+1}[u'][v] - u', v)$ dominates W_i. There exists no vertex set $D \subseteq W_i$ whose representative is u'' such that $c(D) < c(D_{i+1}[u'][v] - u')$ and (D, v) dominates W_i, for otherwise we have $c(D + u') < c(D_{i+1}[u'][v])$ and $(D + u', v)$ dominates W_{i+1}, contradicting the definition of $D_{i+1}[u'][v]$. Thus, $D_{i+1}[u'][v] - u' = D_i[u''][v]$, and we have the claim. ■

From Claim 4, we can compute $D_{i+1}[u][v]$ if $u = u'$.

Lemma 5.

$$D_{i+1}[u'][v] = \min\{D_i[u''][v] + u' \mid u'' \in U_i, y_{u'} < y_{u''}\}. \quad ■$$

Case 2. $u \neq u'$: We further distinguish two cases.

Case 2-1. $y_{u'} < y_u$: In this case, we have that $u' \notin D_{i+1}[u][v]$, for otherwise $y_{u'} < y_u$ implies that the representative of the vertex set is u', contradicting the definition of $D_{i+1}[u][v]$. We have the following.

Lemma 6. *Suppose $w_{i+1} = u' \in U$, and let $u \in U_i$ be a vertex with $y_{u'} < y_u$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u', v) \in E(G), \\ \infty & \text{otherwise.} \end{cases}$$

Proof: Since $u' \notin D_{i+1}[u][v]$ and W_i has no vertex dominating u', v must dominate u'. Thus, $D_{i+1}[u][v]$ does not exist if $(u', v) \notin E(G)$. Assume $(u', v) \in E(G)$. We have $(D_{i+1}[u][v], v)$ dominates W_i by definition. There exists no vertex set $D \subseteq W_i$ whose representative is u such that $c(D) < c(D_{i+1}[u][v])$ and (D, v) dominates W_i, for otherwise $(u', v) \in E(G)$ implies that (D, v) dominates W_{i+1}, contradicting the definition of $D_{i+1}[u][v]$. Thus, $D_{i+1}[u][v] = D_i[u][v]$, and we have the lemma. ■

Case 2-2. $y_{u'} > y_u$: We have the followings.

Claim 7. *Suppose $w_{i+1} = u' \in U$, and let $u \in U_i$ be a vertex with $y_{u'} > y_u$ and $u' \notin D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u', v) \in E(G), \\ \infty & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to that of Lemma 6, and is omitted. ■

Claim 8. *Suppose $w_{i+1} = u' \in U$, and let $u \in U_i$ be a vertex with $y_{u'} > y_u$ and $u' \in D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] - u' = D_i[u][v].$$

Proof: Notice that the representative of $D_i[u][v] + u'$ is still u, since $y_{u'} > y_u$. Since W_i has no vertex dominated by u', $(D_{i+1}[u][v] - u', v)$ dominates W_i. There exists no vertex set $D \subseteq W_i$ whose representative is u such that $c(D) < c(D_{i+1}[u][v] - u')$ and (D, v) dominates W_i, for otherwise we have $c(D + u') < c(D_{i+1}[u][v])$ and $(D + u', v)$ dominates W_{i+1}, contradicting the definition of $D_{i+1}[u][v]$. Thus, $D_{i+1}[u][v] - u' = D_i[u][v]$, and we have the claim. ■

Now, we have the following from Claims 7 and 8.

Lemma 9. *Suppose $w_{i+1} = u' \in U$, and let $u \in U_i$ be a vertex with $y_{u'} > y_u$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u', v) \in E(G), \\ D_i[u][v] + u' & \text{otherwise.} \end{cases}$$

Proof: From the claims, we have $D_{i+1}[u][v] = \min\{D_i[u][v], D_i[u][v] + u'\}$ if $(u', v) \in E(G)$, and $D_{i+1}[u][v] = \min\{\infty, D_i[u][v] + u'\}$ if otherwise. Since the weights of vertices are non-negative, we have the lemma. ■

We can see that Lemmas 5, 6, and 9 prove Lemma 1.

B. Proof of Lemma 2

We will compute $D_{i+1}[u][v]$ from D_i. Recall that $w_{i+1} \in V$, and then, let $v' = w_{i+1}$. We distinguish two cases.

Case 1. $y_{v'} < y_v$: We have the followings.

Claim 10. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} < y_v$ and $v' \notin D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u, v') \in E(G), \\ \infty & \text{otherwise.} \end{cases}$$

Proof: Since $v' \notin D_{i+1}[u][v]$ and u is the representative of $D_i[u][v]$, u must dominate v'. Thus, $D_{i+1}[u][v]$ does not exist if $(u, v') \notin E(G)$. Assume $(u, v') \in E(G)$. We have $(D_{i+1}[u][v], v)$ dominates W_i by definition. There exists no vertex set $D \subseteq W_i$ whose representative is u such that $c(D) < c(D_{i+1}[u][v])$ and (D, v) dominates W_i, for otherwise $(u, v') \in E(G)$ implies that (D, v) dominates W_{i+1}, contradicting the definition of $D_{i+1}[u][v]$. Thus, $D_{i+1}[u][v] = D_i[u][v]$, and we have the claim. ■

Claim 11. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} < y_v$ and $v' \in D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] - v' = D_i[u][v].$$

Proof: Since $y_{v'} < y_v$, we have $N(v') \cap W_i \subseteq N(v) \cap W_i$. It follows that $(D_{i+1}[u][v] - v', v)$ dominates W_i. There exists no vertex set $D \subseteq W_i$ whose representative is u such that $c(D) < c(D_{i+1}[u][v] - v')$ and (D, v) dominates W_i, for otherwise we have $c(D + v') < c(D_{i+1}[u][v])$ and $(D + v', v)$ dominates W_{i+1}, contradicting the definition of $D_{i+1}[u][v]$. Thus, $D_{i+1}[u][v] - v' = D_i[u][v]$, and we have the claim. ∎

Now, we have the following from Claims 10 and 11.

Lemma 12. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} < y_v$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u, v') \in E(G), \\ D_i[u][v] + v' & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to that of Lemma 9, and is omitted. ∎

Case 2. $y_{v'} > y_v$: We have the followings.

Claim 13. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} > y_v$ and $v' \notin D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] = \begin{cases} D_i[u][v] & \text{if } (u, v') \in E(G), \\ \infty & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to that of Claim 10, and is omitted. ∎

Claim 14. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} > y_v$ and $v' \in D_{i+1}[u][v]$. Then,*

$$D_{i+1}[u][v] - v' = D_i[u][v'].$$

Proof: Since $y_{v'} > y_v$, we have $N(v') \cap W_i \supseteq N(v) \cap W_i$. It follows that $(D_{i+1}[u][v] - v', v')$ dominates W_i. There exists no vertex set $D \subseteq W_i$ whose representative is u such that $c(D) < c(D_{i+1}[u][v] - v')$ and (D, v') dominates W_i, for otherwise we have $c(D + v') < c(D_{i+1}[u][v])$ and $(D + v', v)$ dominates W_{i+1}, contradicting the definition of $D_{i+1}[u][v]$. Thus, $D_{i+1}[u][v] - v' = D_i[u][v']$, and we have the claim. ∎

Now, we have the following from Claims 13 and 14.

Lemma 15. *Suppose $w_{i+1} = v' \in V$, and let $v \in V_{i+1}$ be a vertex with $y_{v'} < y_v$. Then,*

$$D_{i+1}[u][v] = \begin{cases} \min\{D_i[u][v], D_i[u][v'] + v'\} & \text{if } (u, v') \in E(G), \\ D_i[u][v'] + v' & \text{otherwise.} \end{cases}$$

Proof: The proof is similar to that of Lemma 9, and is omitted. ∎

We can see that Lemmas 12 and 15 prove Lemma 2.

IV. Induced Matching Problem

A *matching* of a graph G is a set of edges no two of which share a common vertex in G. An *induced matching* of G is a matching that is also an induced subgraph. The (*weighted*) *induced matching problem* is to find an induced matching with maximum cardinality (weight) in a given (edge-weighted) graph. The induced matching problem has applications for wireless networks, and recently attracted much attention (See [12] for details). Previous works of the problem for graphs related to ORGs can be found in [12].

In this section, we show that the weighted induced matching problem can be solved in $O(m^6)$ time for ORGs.

An edge set $F \subseteq E(G)$ of a graph G is called an *edge-asteroidal set* [3] if for every edge $(u, v) \in F$, the edges in $F \setminus \{(u, v)\}$ are contained in the same connected component of the graph obtained from G by deleting all vertices in $N[u]$ and $N[v]$. Notice that every edge-asteroidal set is an induced matching by definition. The maximum cardinality of an edge-asteroidal set in G is called the *asteroidal index* of G. The following is shown in [3].

Theorem I. *The weighted induced matching problem for graphs with asteroidal index at most s can be solved in $O(m^{s+2})$ time.* ∎

We have the following for ORGs. It should be noted that the following lemma is implicit in [6], [8].

Lemma 16. *The asteroidal index of any ORG is at most* 4.

Proof: Let G be an ORG with a bipartition (U, V) and an orthogonal ray representation $\mathcal{R}(G) = \{R_w \mid w \in V(G)\}$. Notice that for the representation $\mathcal{R}(G)$, each edge of G can be classified into four types as up-right, down-right, up-left, or down-left, depending on the orientations of the horizontal ray (rightward or leftward) and the vertical ray (upward or downward) corresponding to the endvertices of the edge.

Now, we prove the lemma by contradiction. Suppose contrary that G has an edge-asteroidal set F of size at least 5. Since $|F| \geq 5$, at least two edges in F have the same type. We assume without loss of generality that edges e and f are both of type up-right. For an edge $(u, v) \in E(G)$ of type up-right with $u \in U$ and $v \in V$, two rays R_u and R_v divide the plane into two regions, and we refer to the region above R_u and to the right of R_v as the *inner* region of (u, v), and the other as the *outer* region. Since no pairs of edges in F are joined by an edge, we can further assume that the rays corresponding to the endvertices of e lie in the inner region of f, and we have that the rays corresponding to the endvertices of the other edges in F lie in the outer region of f. Hence, any path from e to any other edge in F must have a vertex adjacent to at least one of the endvertices of f, contradicting to the definition of edge-asteroidal set. Thus, G contains no edge-asteroidal sets of size at least 5. ∎

Thus, we have the following from Theorem I and Lemma 16.

Theorem 17. *The weighted induced matching problem can be solved in $O(m^6)$ time for ORGs.* ∎

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